Singapore Physics Olympiad Training

2024 Selection Test

for the Asian and International Physics Olympiads

a. This is a **4 hour** test. Attempt all questions. The maximum total score is **85**; marks allocated for each question part are indicated in square brackets.

b. Check that there are a total of **11 printed pages** (*including* this cover page). The last page contains a table of physical constants that you may refer to and use.

c. Begin your answer for each question on a **fresh sheet of paper**, and present your working and answers clearly. Your answer sheets should be sorted according to the order of the questions.

d. Write your name on the **top right hand corner of every answer sheet** you submit.

e. Please **complete and sign the declaration on page 2**, which should be **stapled together and submitted** with your answer sheets.

f. You may use a standard (non-programmable) scientific **calculator** in accordance with the statutes of the International Physics Olympiad.

g. No books or documents relevant to the test may be brought into the examination room.

Declaration

I declare that I will be fully committed to the training for and participation in the Asian Physics Olympiad and/or the International Physics Olympiad if selected. I will check first with the MOE coordinator before taking on **additional commitments not listed below**.

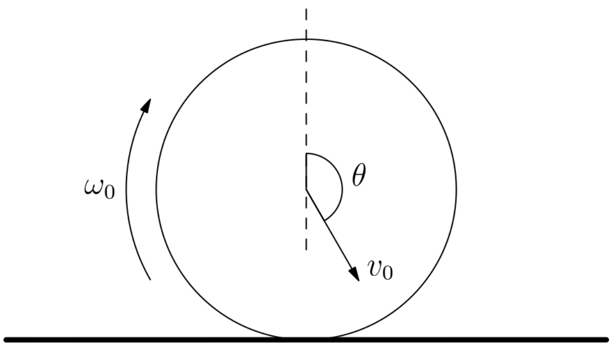
Potential limitations to my commitment in the period from now to end-July 2024 are **described exhaustively** in the box below, such as other academic competitions, CCA commitments (school-related or otherwise), travel plans, etc.

Name and signature: ____________________

Question:	1	2	3	4	5	6	7	8	9	Total
Points:	5	5	7	6	20	10	9	10	13	85
Score:										

Total time: 4 hours

1. A thin uniform ring of mass m falls onto a rough floor. The initial velocity of the centre of mass is v_0, at an angle θ clockwise from the upwards vertical when it contacts the floor (refer to the diagram). It is also rotating with angular velocity ω_0 clockwise. The ground is rough enough so that the ring achieves no-slipping right after it contacts the ground. Denote the coefficient of restitution as e and the gravitational acceleration as g.



   (a) Find the velocity of the ring after the first bounce, and its angular velocity. [4]

   (b) Suppose the ring bounces straight up after touching the ground. Find the maximum height reached by the ring. [1]

2. Two square plates of side length L, constructed from an ideal conducting material, are separated by an air gap of h. Both plates are parallel to and have the same projection onto the xy-plane. The space between them is permeated with a magnetic field B which is parallel to the x-axis. A metal rod of mass m, length h and resistance R is placed parallel to the z-axis at the maximum y-position such that it is just touching both plates and allowed to fall from rest until time T, when it reaches the minimum y-position and loses contact with both plates. Assume that gravity acts in the negative y-direction and that the rod remains in contact with both plates for as long as possible.

   (a) Derive the expression for the velocity v of the rod. [4]

   (b) Describe and explain qualitatively the behaviour of the rod after a long time but before time T, assuming T is very large. [1]

3. Consider a magnetic monopole at the origin emitting a magnetic field $\mathbf{B}(\mathbf{r}) = \frac{\mu}{r^3}\mathbf{r}$. The monopole is fixed. An electron of charge $e = -1.6 \times 10^{-19}$ C and mass m is at position $\mathbf{r}$, moving with velocity v. In addition there is an arbitrary radially symmetric potential field $U(\mathbf{r})$ acting on the electron, this is generally to ensure that the electron's path would be bounded. (Hint: $U(\mathbf{r})$ should not appear in your answers for (b) and (c))

   (a) Write down the equation of motion for the electron. (Hint: write the equation out in the vector form) [1]

   (b) By considering the rate of change of orbital angular momentum $\mathbf{L}$ and the quantity [4]

$$\mathbf{S} = -e\mu\frac{\mathbf{r}}{r}$$

   Prove that the quantity $\mathbf{J} = \mathbf{L} + \mathbf{S}$ is conserved. This can be interpreted as "total" angular momentum, and $\mathbf{S}$ can be interpreted as a spin angular momentum associated with the energy field of the system.

   (c) By considering the component of total angular momentum $\mathbf{J}$ in the radial direction $\hat{r} = \frac{\mathbf{r}}{r}$, show that the angle between the two vectors are constant. Hence describe the surface that the path of the electron must lie on, and sketch some possible paths. [2]

4. A mass is attached to the end of a massless rod of length l, which is then raised to near-vertical then released. Let the angle between the rod and the vertical be $\epsilon \ll 1$.

   (a) For motion between $\epsilon \ll \theta_0 \ll 1$, find the equation of motion of the pendulum. Hence, find the time taken to reach θ_0, and the angular velocity when it reaches θ_0. [4]

   (b) By considering the motion of the pendulum past θ_0, justify that the period of the pendulum T tends to $4\sqrt{\frac{l}{g}}\ln\frac{1}{\epsilon}$ as $\epsilon \to 0$ [2]

5. The 1997 Nobel Prize in Physics was awarded to Steven Chu, Claude Cohen-Tannoudji and William D. Phillips for development of methods to cool and trap atoms with laser light. The laser light cools the particles down to the µK temperature range, allowing particles to move at speeds less than 1 km/h. At these speeds, scientists can study individual atoms with great accuracy, opening the gateway to a deeper understanding of the properties of gases at low temperatures.

In this problem, you will be analysing the basics of laser cooling and a method of cooling using the Doppler effect.

Part A: Radiation Pressure

An atom can be represented as an electron of charge q and mass m attached to a spring with spring constant $m\omega_0^2$, experiencing a damping force proportional to its speed with proportionality constant $m\gamma$. It is driven by an electromagnetic field $E = E_0 \cos\omega t$, whose frequency is very close to the resonant frequency of the atom: defining the detuning $\Delta = \omega - \omega_0$, we may write $|\Delta| \ll \omega$. Also assume that $\gamma \ll \omega$.

(a) Write down the differential equation satisfied by the displacement $x(t)$ of the electron from its equilibrium position. Hence, find x_0 and ϕ in terms of $q, m, \Delta, \gamma, E_0$, and ω_0. [4]

(b) Find the average power $\langle P \rangle$ absorbed over one cycle of the electromagnetic field, expressing your answer in terms of q, m, γ, E_0 and Δ. [2]

The saturation intensity I_s is the intensity of the laser beam which causes the atom to spend one-quarter of its time in the excited state, and it is a quantity which appears widely in literature on laser cooling:

$$I_s = \frac{\varepsilon_0 m c \gamma^2 \hbar\omega}{q^2}.$$

(c) Show that the average rate R at which photons are absorbed by an atom is [1]

$$R = \frac{I/I_s}{1 + 4\Delta^2/\gamma^2}\gamma$$

(d) Find an expression for the force on an atom due to resonant absorption. [1]

Part B: Doppler Cooling

For gases to reach low temperatures, their atoms must achieve low velocities as well. This damping mechanism is different from the radiation damping γ described earlier, and relies on the Doppler effect on an atom's interaction with its surrounding electromagnetic field.

Consider the case where the atom is moving in one dimension (x), and the electromagnetic field is propagating in the $+x$-direction with angular frequency ω.

(e) Write down the angular frequency ω' the particle sees the field oscillating at while it is moving at velocity v. Show that for $v \ll c$, the Doppler shift $\delta\omega_D = \omega' - \omega$ can be expressed as $-kv$, where k is the wave number of the electromagnetic wave. [2]

This means that when you calculate the force, you can take into account the Doppler effect by replacing Δ with $\Delta + \delta\omega_D$.

It is clear that when an atom travels into an incoming laser beam, it will slow down. However, if the radiative forces continue to act, it will accelerate in the opposite direction. We want the atom to experience no further force after coming to a stop. This will be possible if we illuminate the atom with two identical laser beams propagating in opposite directions.

(f) Calculate the net force F on an atom moving with velocity v. Express your answer in terms of quantities defined earlier. Furthermore, show that in the low-velocity limit ($kv \ll |\Delta|$ and γ), the net force F can be written as $F = -\alpha v$, where α is to be determined. [3]

Part C: Heating Due to Photon Recoil

The damping force causes the atom to slow down. However, there is another heating mechanism – the atom absorbs a photon from the electromagnetic field, making a transition from the ground state to the excited state. The excited state is not stable, so the atom returns to the ground state by emitting a photon in a random direction (+ or – in one dimension). In both instances, momentum is not transferred to the atom in a continuous manner, but rather in units of $\hbar k$.

This quantized gains and losses of energy causes the momentum of the atom to take the path of a random walk. In a given time interval, the number of steps is the number of photons absorbed and emitted. Take the probability of absorption of a photon from either beam to be equally likely. Thus, each absorption and emission results in two steps of the random walk. In a time interval dt, the atom executes $dN = 2R_{\text{tot}}\, dt$ steps, where $R_{\text{tot}} = R_+ + R_-$ is the total absorption rate from the two beams.

For this one-dimensional walk, the average momentum remains zero, but the RMS momentum equals the square root of the number of steps times the step size:

$$\sqrt{\langle p^2 \rangle} = \sqrt{N}\hbar k.$$

(g) Calculate the rate at which the atom's energy increases due to the heating. [2]

(h) Find the equilibrium temperature T associated with this atomic motion. [3]

(i) Determine the resulting minimum temperature $T_{\min}$ and the corresponding Δ when this is achieved. [2]

6. An electron is confined to move along the circumference of a thin ring with radius r.

   (a) Find the allowed values of its kinetic energy, giving your answer in terms of the electron mass m and Planck's constant h. [2]

   The ring is now placed in a constant (but not necessarily uniform) magnetic field directed into the page, such that the magnetic flux through the ring is Φ.

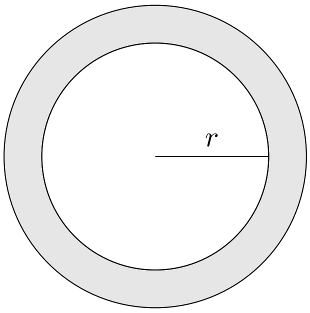


Figure 1: Illustration of the ring.

   (b) By considering the energy contribution from the current of the moving charge, show that the total energy of the system (up to an additive constant) is given by [2]

$$E = \frac{p_{\text{eff}}^2}{2m} = \frac{1}{2m}\left(p + \frac{e\Phi}{2\pi r}\right)^2$$

   where p is the electron's momentum (treating anticlockwise as positive) and $-e$ is the charge of the electron.

   (c) Treating p_{eff} as the total effective momentum of the electron's quantum wave, find the magnitude and direction of the current flowing in the ring in the ground state(s) and first excited state(s) of the electron's kinetic energy when $\Phi = \frac{h}{2e}$. [2]

   A typical electron double-slit experiment is set up as shown in the diagram below, with a solenoid placed just behind the two slits. The width of each slit is small but finite. The velocity of the electron beam is v, the distance between the slits is d, and the distance from the slits to the screen is $L \gg d$. The magnetic field of the solenoid is directed into the page, and the total magnetic flux through the solenoid is $\Phi = \frac{Nh}{2e}$, where N is the number of coils in the solenoid.

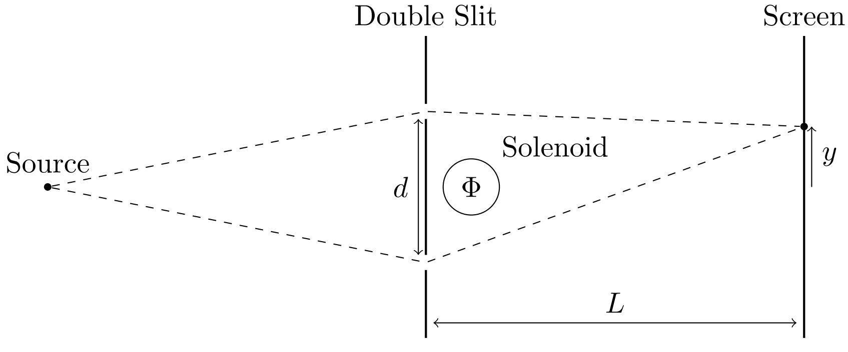
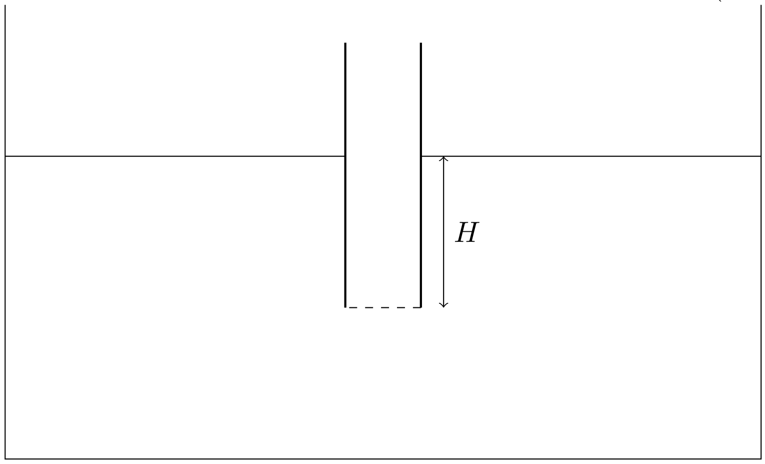


Figure 2: Illustration of the electron double-slit experiment.

   (d) Sketch the intensity of electrons detected as a function of the vertical position y along the screen. Include the distance between extrema in the sketch. [4]

7. A straw with its bottom end covered is placed in a large tank of water such that its bottom end is H below the surface of the water. At $t = 0$ the barrier (dashed) vanishes.



(a) Use Bernoulli's principle to find the total time it takes for the water to reach the surface level inside the tube. Explain why this value may be inaccurate. [2]

(b) The Navier-Stokes equation (1) can be used to obtain a more accurate solution. [2]

$$\frac{\partial \vec{\mathbf{u}}}{\partial t} + (\vec{\mathbf{u}} \cdot \nabla)\vec{\mathbf{u}} = -\frac{\nabla P}{\rho} + \vec{\mathbf{g}} \tag{1}$$

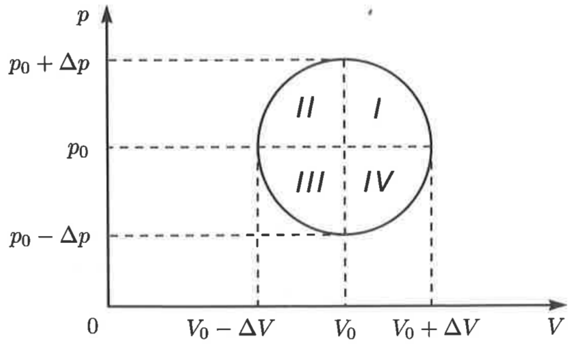
Assuming irrotational flow such that $\vec{\mathbf{u}} = \nabla\phi(x, y, z, t)$. Show that the equation reduces to (2) where C is a constant. (Hint: You may want to use the fact that $\vec{\mathbf{A}} \times (\nabla \times \vec{\mathbf{A}}) = \frac{1}{2}\nabla A^2 - (\vec{\mathbf{A}} \cdot \nabla)\vec{\mathbf{A}}$ for any vector field $\vec{\mathbf{A}}$.)

$$\frac{\partial \phi}{\partial t} + \frac{u^2}{2} + \frac{P}{\rho} + gz = C \tag{2}$$

(c) Find the function ϕ for the region inside the straw in terms of velocity of the water surface at the top of the straw and z (the vertical distance from the bottom of the straw). Explain how you arrived at the answer. [2]

(d) Using (2) determine the maximum height that the water can reach above the surrounding water level outside the straw. (Hint: The substitution $r = \frac{v_z^2}{2}$ may be useful) [3]

8. A cyclic process with one mole of an ideal polyatomic gas appears as a circle on the pressure-volume ($P-V$) diagram. Coordinates of the circle centre are (P_0, V_0), the diameter along the pressure axis is $2\Delta P$, and the diameter along the volume axis is $2\Delta V$.



(a) Determine all pairs of diametrically opposite points of the circle (P, V) with equal heat capacities. Calculate these heat capacities in terms of known quantities, C_V, C_p and R. [6]

(b) Compare heat capacities of two arbitrary diametrically opposite points lying in quadrants 2 and 4 of the circle. Which of these points has greater heat capacity? Why? [1]

(c) Form a pair of simultaneous (algebraic) equations that you would use to determine the values (P, V) where entropy is maximum and minimum during the cycle. Comment, with mathematical justification, whether these points are diametrically opposite. [3]

9. From special relativity, it is known that energy and mass are equivalent and interchangeable. Some of the results of general relativity can be obtained by treating the gravitational mass as $m_g = \frac{F}{c^2}$, where F is the total non-potential energy of the particle.

   (a) Consider a photon fired radially outwards from a large mass M. If the frequency received by an observer infinitely far away is f_0, determine its frequency $f(r)$ as a function of the radial distance $r \gg \frac{GM}{c^2}$ away from the large mass. [2]

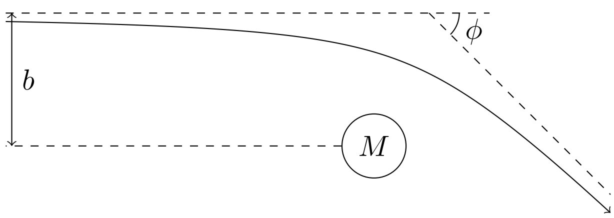
   (b) Hence, find the effective Lorentz factor $\gamma_g(r)$ by which time and length are dilated and contracted with respect to an observer at infinity, and determine the speed v at which a non-accelerating frame would experience the same effect. [1]

   To account for these effects, under weak gravity $\left(r \gg \frac{GM}{c^2}\right)$, the usual invariant proper time interval can be modified to

$$d\tau^2 = \left(1 - \frac{2GM}{rc^2}\right) dt^2 - \frac{1}{c^2}\left[\left(1 + \frac{2GM}{rc^2}\right) dr^2 + r^2 d\theta^2\right]$$

   where the coordinates (r, θ) are the usual polar coordinates with mass M at the origin, and all coordinates are taken with respect to an observer at infinity.

   A particle of mass m is fired towards an object of mass M from very far away with impact parameter $b \gg \frac{GM}{c^2}$ and initial velocity u, such that the particle's trajectory is deflected by an angle $\phi \ll 1$.



   (c) Show that the total energy of the particle is given by [3]

$$E^2 = \frac{c^2}{\alpha^2}\left(m^2c^2 + \alpha^2 p_r^2 + r^2 p_\theta^2\right)$$

   where $p_x = m\frac{dx}{d\tau}$ is the x-component of the particle's momentum and $\alpha = 1 + \frac{GM}{rc^2}$.
   [*Hint: If* $ds^2 = A\ dx^2 + B\ dy^2$, *then* $\vec{a} \cdot \vec{b} = Aa_xb_x + Ba_yb_y$.]

   (d) Show that this effectively reduces to an additional central force acting on the particle of the form [3]

$$\vec{F} = \frac{d\vec{p}}{d\tau} = -\frac{\beta}{r^4}\hat{r}$$

   where β is some constant you should determine.

   (e) Hence or otherwise, determine the angle of deflection ϕ to leading order in $\frac{GM}{bc^2}$ and compare your results for a massive particle ($u \ll c$) and a photon ($u = c$) to the classical case ($\phi = \frac{2GM}{bu^2}$). [4]
   You may make use of the following integral without proof:

$$\int_{-\infty}^{\infty} \frac{dx}{(x^2+1)^k} = \begin{cases} 2 & (k = \frac{3}{2}) \\ \frac{4}{3} & (k = \frac{5}{2}) \end{cases}$$

2018 CODATA adjustment From: **http://physics.nist.gov/constants**

Fundamental Physical Constants — Frequently used constants

Quantity	Symbol	Value	Unit	Relative std. uncert. u_r
speed of light in vacuum	c	$299\,792\,458$	$\mathrm{m\ s^{-1}}$	exact
Newtonian constant of gravitation	G	$6.674\,30(15)\times10^{-11}$	$\mathrm{m^3\ kg^{-1}\ s^{-2}}$	2.2×10^{-5}
Planck constant*	h	$6.626\,070\,15\times10^{-34}$	$\mathrm{J\ Hz^{-1}}$	exact
	$\hbar$	$1.054\,571\,817\ldots\times10^{-34}$	J s	exact
elementary charge	e	$1.602\,176\,634\times10^{-19}$	C	exact
vacuum magnetic permeability $4\pi\alpha\hbar/e^2c$	μ_0	$1.256\,637\,062\,12(19)\times10^{-6}$	$\mathrm{N\ A^{-2}}$	1.5×10^{-10}
vacuum electric permittivity $1/\mu_0c^2$	ϵ_0	$8.854\,187\,8128(13)\times10^{-12}$	$\mathrm{F\ m^{-1}}$	1.5×10^{-10}
Josephson constant $2e/h$	K_J	$483\,597.848\,4\ldots\times10^{9}$	$\mathrm{Hz\ V^{-1}}$	exact
von Klitzing constant $\mu_0c/2\alpha=2\pi\hbar/e^2$	R_K	$25\,812.807\,45\ldots$	Ω	exact
magnetic flux quantum $2\pi\hbar/(2e)$	Φ_0	$2.067\,833\,848\ldots\times10^{-15}$	Wb	exact
conductance quantum $2e^2/2\pi\hbar$	G_0	$7.748\,091\,729\ldots\times10^{-5}$	S	exact
electron mass	m_e	$9.109\,383\,7015(28)\times10^{-31}$	kg	3.0×10^{-10}
proton mass	m_p	$1.672\,621\,923\,69(51)\times10^{-27}$	kg	3.1×10^{-10}
proton-electron mass ratio	$m_\mathrm{p}/m_\mathrm{e}$	$1836.152\,673\,43(11)$		6.0×10^{-11}
fine-structure constant $e^2/4\pi\epsilon_0\hbar c$	α	$7.297\,352\,5693(11)\times10^{-3}$		1.5×10^{-10}
inverse fine-structure constant	α^{-1}	$137.035\,999\,084(21)$		1.5×10^{-10}
Rydberg frequency $\alpha^2m_\mathrm{e}c^2/2h$	cR_∞	$3.289\,841\,960\,2508(64)\times10^{15}$	Hz	1.9×10^{-12}
Boltzmann constant	k	$1.380\,649\times10^{-23}$	$\mathrm{J\ K^{-1}}$	exact
Avogadro constant	N_A	$6.022\,140\,76\times10^{23}$	$\mathrm{mol^{-1}}$	exact
molar gas constant $N_\mathrm{A}k$	R	$8.314\,462\,618\ldots$	$\mathrm{J\ mol^{-1}\ K^{-1}}$	exact
Faraday constant $N_\mathrm{A}e$	F	$96\,485.332\,12\ldots$	$\mathrm{C\ mol^{-1}}$	exact
Stefan-Boltzmann constant $(\pi^2/60)k^4/\hbar^3c^2$	σ	$5.670\,374\,419\ldots\times10^{-8}$	$\mathrm{W\ m^{-2}\ K^{-4}}$	exact
		Non-SI units accepted for use with the SI		
electron volt (e/C) J	eV	$1.602\,176\,634\times10^{-19}$	J	exact
(unified) atomic mass unit $\frac{1}{12}m(^{12}\mathrm{C})$	u	$1.660\,539\,066\,60(50)\times10^{-27}$	kg	3.0×10^{-10}

* The energy of a photon with frequency ν expressed in unit Hz is $E=h\nu$ in J. Unitary time evolution of the state of this photon is given by $\exp(-iEt/\hbar)|\varphi\rangle$, where $|\varphi\rangle$ is the photon state at time $t=0$ and time is expressed in unit s. The ratio $Et/\hbar$ is a phase.