Singapore Physics Olympiad Training

2023 Selection Test

for the Asian and International Physics Olympiads

a. This is a **four-hour** test. Attempt all questions. The maximum total score is **75**; marks allocated for each question part are indicated in square brackets.

b. Check that there are a total of **9 printed pages** (*including* this cover page). The last page contains a table of physical constants that you may refer to and use.

c. Begin your answer for each question on a **fresh sheet of paper**, and present your working and answers clearly. Your answer sheets should be sorted according to the order of the questions.

d. Write your name on the **top right hand corner of every answer sheet** you submit.

e. Please **complete and sign the declaration on page 2**, which should be **stapled together and submitted** with your answer sheets.

f. You may use a standard (non-programmable) scientific **calculator** in accordance with the statutes of the International Physics Olympiad.

g. No books or documents relevant to the test may be brought into the examination room.

Declaration

I declare that I will be fully committed to the training for and participation in the Asian Physics Olympiad and/or the International Physics Olympiad if selected. I will check first with the MOE coordinator before taking on **additional commitments not listed below**.

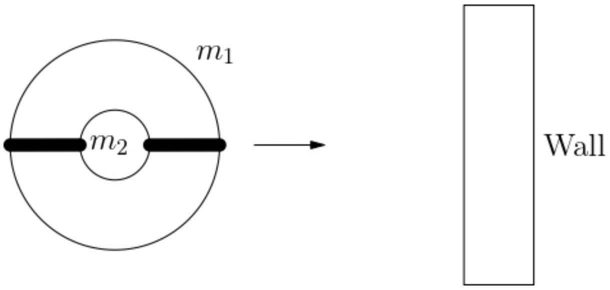
Potential limitations to my commitment in the period from now to end-July 2023 are **described exhaustively** in the box below, such as other academic competitions, CCA commitments (school-related or otherwise), travel plans, etc.

Name and signature: ______________________________

Question:	1	2	3	4	5	6	7	8	Total
Points:	5	9	10	9	7	6	10	19	75
Score:									

1. We model the collision of a compound object with a rigid vertical wall. The object is made up of a spherical shell of mass m_1 that is joined by a horizontal rod to the centre of an inner ball of mass m_2.

   The rod has negligible mass and an effective spring constant k, such that the magnitude of the restoring force is $F = kx$ when the distance between the centres of the masses is x. The rod does not twist or flex, but can compress and stretch as the masses are displaced from their initially concentric positions.



   The object moves with constant horizontal velocity v_i directly towards the wall, colliding with it. Suppose that $m_1 > m_2$. Ignore any vertical forces, and suppose the object does not spin or rotate. Assume that all collisions are elastic.

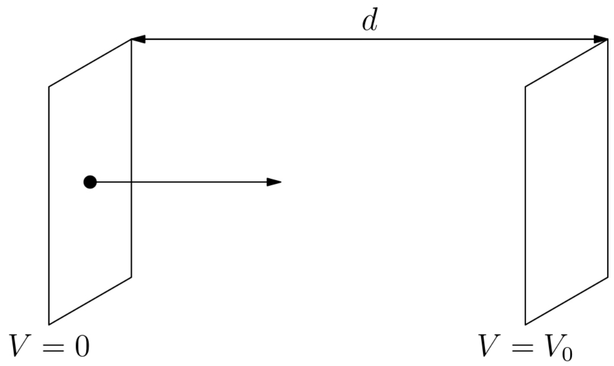
   (a) Derive an expression for the coefficient of restitution $e \equiv v_f/v_i$, where v_i and v_f are the initial and final speeds of the centre of mass of the object. [1]

   (b) Show that after the collision, the masses m_1 and m_2 oscillate about their centre of mass in simple harmonic motion. [2]

   (c) Find the angular frequency ω of oscillation and maximum distance X between the centres of masses, in terms of v_i, m_1, m_2, and k. [2]

Q1 total: 5

2. This question is about thermionic emission. Consider two very large parallel plates, each of area A, separated by a distance d. Electrons are emitted from rest from the hot cathode at potential $V = 0$, and accelerated across a gap to the anode at potential $V = V_0$ as shown in the figure.



   The moving electrons, termed as space charge, build up to the point where the electric field at the surface of the cathode is zero, with a steady current I flowing between the plates.

   Suppose the plates are large compared to the separation between them (i.e. $A \gg d^2$). Defining x to be the distance from the cathode, the charge density[1] ρ and speed of moving

[1]To be clear, we mean the charge density "per unit volume".

electrons v are both functions of x. Denote the charge and mass of an electron to be $-q$ and m respectively.

(a) Find the relationship between the potential V and the speed v at distance x from the cathode. [1]

(b) At steady state, the current I is independent of x. Show that V obeys the following differential equation as a function of x: [3]

$$\frac{d^2V}{dx^2} + \frac{I}{\varepsilon_0 A}\sqrt{\frac{m}{2q}}V^{-\frac{1}{2}} = 0.$$

(c) Hence find the potential V as a function of x, in terms of I, m, A, q, and other fundamental constants. [3]

(d) Find the relationship between the steady current I and the applied potential difference V_0, in terms of the geometry of the plates and fundamental constants. [2]

Q2 total: 9

3. A stationary observer, Alice, observes a proton P_1 in a magnetic field with flux density $B_z = 1.0\,\text{T}$ in the $+z$-direction. P_1 moves in a circle of radius R_1 in the xy-plane with speed $v = 3.00 \times 10^5\,\text{m/s}$.

(a) State whether P_1 moves clockwise or anticlockwise (when looking down at the proton). [1]

(b) Find the radius R_1 of the circle, and find the time the proton takes to complete one circle. [2]

Relative to Alice, another observer Bob moves with constant velocity $v_B = 1.00 \times 10^4\,\text{m/s}$ in the $+x$-direction.

(c) Describe the motion of P_1 in Bob's frame with a sketch. Remember to indicate the orientation of your axes. [1]

Another proton P_2 is at rest in Alice's frame, at a distance of $1.00\,\text{cm}$ from the centre of the circle described above, which we assume is sufficiently far away that the interaction between the two protons can be neglected.

(d) Describe the motion of P_2 in Bob's frame with a sketch. Remember to indicate the orientation of your axes. [1]

(e) Bob attributes the motion of P_1 and P_2 to electromagnetic fields in his frame. What static electric and magnetic fields could result in the motion of both protons as observed by him? [2]

Another observer Carol has a stationary position in Alice's frame and sees P_1 as stationary. How can this be?

(f) Describe the motion of P_2 in Carol's frame with a sketch. [1]

(g) Carol attributes the motion of P_2 to electromagnetic fields in her frame. What static electric and magnetic fields could result in the motion of both protons as observed by her? [2]

Q3 total: 10

4. The energy transferred by an electromagnetic wave per unit time per unit surface area is given by the Poynting vector

$$\mathbf{S} = \frac{1}{\mu_0}\mathbf{E} \times \mathbf{B},$$

where the direction of the vector $\mathbf{S}$ is the direction of energy transfer.

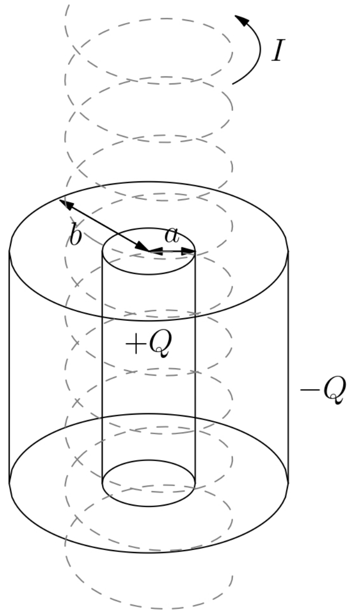
(a) Show the volume density of the linear momentum of an electromagnetic wave is [2]

$$\mathbf{p}_V = \frac{1}{c^2\mu_0}\mathbf{E} \times \mathbf{B},$$

where c is the speed of light.

We now consider a "paradox" regarding the conservation of angular momentum. Two long, coaxial cylindrical shells, shown in the figure, both have length l. The inner one has radius a and electric charge $+Q$ uniformly distributed along its surface, while the outer one has radius $b > a$ and electric charge $-Q$ uniformly distributed over its surface.

The cylinders are made of the same material, having mass per unit area equal to σ. Between them, there is another long solenoid with radius R $(a < R < b)$ that is also coaxial with the two cylinders. The solenoid has n turns per unit length and carries an electric current I.



The solenoid is held fixed in space, but the cylindrical shells can freely and independently rotate around their common axis. Initially, all parts of the system are at rest. When the current in the solenoid is gradually reduced to zero, the cylinders begin to rotate.

(b) Find the final angular velocities of each of the cylinders, giving their magnitude and orientation. [4]

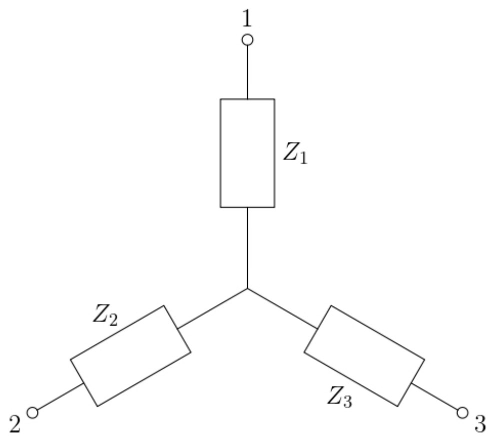
(c) Since no external force acts on the system, the net angular momentum should be conserved. Where did the extra angular momentum come from? Support your answer with equations. [3]

Q4 total: 9

5. Three circuit elements are connected to a central junction in a "star" shape, as shown in the figure. One is a resistor, one is an inductor, and one is a capacitor, although it is not known which is which.

A physicist connects an AC source with fixed voltage V_s across a pair of terminals, at the same time connecting an AC voltmeter to one of the terminals (the other end of the voltmeter is always fixed at the central junction). She obtains the following readings:

(a) Determine the value of V_s (i.e. the reading on the AC voltmeter when it is hooked up directly to the AC source). [3]



AC source terminals	AC voltmeter terminal	Voltmeter reading
1 & 2	1	20.8 V
1 & 2	2	15.6 V
1 & 3	1	24.0 V
1 & 3	3	10.0 V
2 & 3	2	58.5 V
2 & 3	3	32.5 V

(b) Determine the possible identities of Z_1, Z_2, Z_3 (i.e. which is the resistor, inductor, capacitor). [1]

(c) Now, an AC ammeter is also connected in series with the AC source. Find the ratio of currents $I_{12} : I_{13} : I_{23}$, where I_{ij} denotes the value on the AC ammeter when the AC source is connected to terminals i and j. [3]

Q5 total: 7

6. This question is about deriving thermodynamic identities.

(a) Consider a mole of ideal gas at pressure P, volume V, and temperature T. Denote the heat capacity at constant volume as C_V. Show that the heat capacity C is given by [2]

$$C = C_V + \frac{R}{1 + \frac{V}{P}\frac{dP}{dV}}.$$

(b) Consider two ideal gases A and B which are mixed. There are n_1 moles of gas A and n_2 moles of gas B, and the molar heat capacities at constant volume of gases A and B are c_{v1} and c_{v2} respectively. Find the adiabatic constant of the mixed gas. [4]

Q6 total: 6

7. Weather balloons float at high altitudes and need to withstand very low temperatures and pressures. When inflated, the effective radial tension on the surface of a spherical balloon of radius r is given by

$$F_T = 16\pi r_0 \kappa RT \left(\lambda - \frac{1}{\lambda^5}\right),$$

where r_0 is the radius of the balloon when there is no tension, $\lambda \equiv r/r_0$ is the size inflation ratio, κ is a constant with dimensions of inverse area, R is the molar gas constant, and T is the temperature of the air.

(a) Find the pressure difference Δp between the air inside and outside the balloon, giving your answer in terms of λ, T, r_0 and other constants. [2]

(b) Determine the maximum pressure difference Δp_m as a function of T and the radius r_m in terms of r_0 that achieves this value. [3]

The constant κ can be determined from the amount of gas required to inflate a balloon. Suppose at ground level, the temperature is $T_0 = 290\,\text{K}$ and the pressure is $p_0 = 1.01 \times 10^5\,\text{Pa}$.

An unstretched balloon of radius r_0 contains $n_i = 10\,\text{mol}$ of helium. After the balloon is pumped with helium so that it contains a total of $n_f = 40\,\text{mol}$ of helium, the balloon has a radius $r = 1.5r_0$.

(c) Determine the value of κ for this balloon. [5]

Q7 total: 10

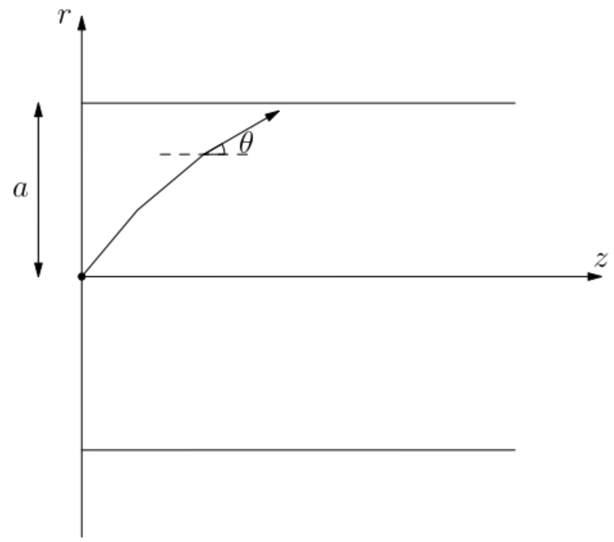
8. This question is on pulse spreading in fibre optics.

Consider a cylindrical optical fibre in the region $0 \le r \le a$ for $z > 0$, see diagram. There is a light source at $r = z = 0$ that emits monochromatic waves. The refractive index $n(r)$ is a function of the radial distance from the cylindrical axis.

Along the path of a ray, if the refractive index at some point is n and the angle the ray makes with the horizontal (z-axis) is θ, we may use Snell's Law to conclude that

$$n \cos\theta = \tilde{\beta}$$

is a constant at all points along the path of the ray.



(a) Show that the path that a ray takes satisfies [2]

$$\frac{d^2 r}{dz^2} = \frac{1}{2\tilde{\beta}^2} \frac{d(n(r)^2)}{dr}.$$

This is known as the ray equation.

(b) The fibre is characterised by the following refractive index distribution:

$$n(r)^2 = n_1^2 \left[1 - 2\Delta \left(\frac{r}{a}\right)^2\right], \qquad 0 \le r \le a$$

where $\Delta \ll 1$ and n_1 are constants.

The refractive index of the medium outside the optical fibre is uniform, with the value n_2 given by

$$n(r)^2 = n_2^2 = n_1^2(1-2\Delta), \qquad r > a.$$

The initial angle of projection θ_1 has to be small enough for the ray to return to the z-axis.

i. Assuming this is the case, find the equation of the path $r = r(z)$ taken by the ray of light, as well as the position z_1 of the first instance that the ray returns to the z-axis. Express your answers in terms of n_1, Δ, a, and $\tilde{\beta}$. [4]

ii. If $\theta_1 \ll 1$ such that we make the approximation $\cos\theta_1 \approx 1$, state the value of z_1. [1]

iii. Find the maximum possible value of θ_1, in terms of Δ. [1]

(c) One of the important characteristics of a waveguide is pulse dispersion, the temporal spreading of a pulse of light launched into the waveguide. This is due to the difference in time taken by different rays. To calculate this dispersion, we calculate the time taken by a ray to traverse a given length of the waveguide. [2]

Define the maximum radial distance the ray reaches from the z-axis to be r_t. Show that the time taken for the light ray to first reach a distance r_t from the z-axis is given by

$$\frac{1}{c}\int_0^{r_t} \frac{n(r)^2}{\sqrt{n(r)^2 - \tilde{\beta}^2}} dr$$

where c is the speed of light in vacuum.

(d) For the fibre optic medium described in (b):

i. Find the time taken for a light ray to first reach a distance r_t from the z-axis, expressing your answer in terms of $a, n_1, \tilde{\beta}, \Delta$, and c. [3]

ii. Calculate the difference in the maximum and minimum times for rays to travel a distance z along the z-axis, in terms of n_1, Δ, and c. This time difference τ can be taken to be the pulse dispersion time. [3]

(e) To appreciate the small dispersion given in the previous part, let us consider the pulse dispersion in a cylindrical fibre optic medium with the same physical dimensions but with homogeneous refractive index n_1, while the outside is still kept at refractive index n_2 satisfying $n_2^2 = n_1^2(1-2\Delta)$. [3]

Find the pulse dispersion time over a distance z along the z-axis for such a setup, in terms of n_1, Δ, and c.

Q8 total: 19

2018 CODATA adjustment From: **http://physics.nist.gov/constants**

Fundamental Physical Constants — Frequently used constants

Quantity	Symbol	Value	Unit	Relative std. uncert. u_r
speed of light in vacuum	c	299 792 458	m s^{-1}	exact
Newtonian constant of gravitation	G	$6.674\,30(15) \times 10^{-11}$	m^3 kg^{-1} s^{-2}	2.2×10^{-5}
Planck constant*	h	$6.626\,070\,15 \times 10^{-34}$	J Hz^{-1}	exact
	$\hbar$	$1.054\,571\,817\ldots \times 10^{-34}$	J s	exact
elementary charge	e	$1.602\,176\,634 \times 10^{-19}$	C	exact
vacuum magnetic permeability $4\pi\alpha\hbar/e^2c$	μ_0	$1.256\,637\,062\,12(19) \times 10^{-6}$	N A^{-2}	1.5×10^{-10}
vacuum electric permittivity $1/\mu_0 c^2$	ϵ_0	$8.854\,187\,8128(13) \times 10^{-12}$	F m^{-1}	1.5×10^{-10}
Josephson constant $2e/h$	K_J	$483\,597.848\,4\ldots \times 10^9$	Hz V^{-1}	exact
von Klitzing constant $\mu_0 c/2\alpha = 2\pi\hbar/e^2$	R_K	$25\,812.807\,45\ldots$	Ω	exact
magnetic flux quantum $2\pi\hbar/(2e)$	Φ_0	$2.067\,833\,848\ldots \times 10^{-15}$	Wb	exact
conductance quantum $2e^2/2\pi\hbar$	G_0	$7.748\,091\,729\ldots \times 10^{-5}$	S	exact
electron mass	m_e	$9.109\,383\,7015(28) \times 10^{-31}$	kg	3.0×10^{-10}
proton mass	m_p	$1.672\,621\,923\,69(51) \times 10^{-27}$	kg	3.1×10^{-10}
proton-electron mass ratio	m_p/m_e	$1836.152\,673\,43(11)$		6.0×10^{-11}
fine-structure constant $e^2/4\pi\epsilon_0\hbar c$	α	$7.297\,352\,5693(11) \times 10^{-3}$		1.5×10^{-10}
inverse fine-structure constant	α^{-1}	$137.035\,999\,084(21)$		1.5×10^{-10}
Rydberg frequency $\alpha^2 m_e c^2/2h$	cR_∞	$3.289\,841\,960\,2508(64) \times 10^{15}$	Hz	1.9×10^{-12}
Boltzmann constant	k	$1.380\,649 \times 10^{-23}$	J K^{-1}	exact
Avogadro constant	N_A	$6.022\,140\,76 \times 10^{23}$	mol^{-1}	exact
molar gas constant $N_A k$	R	$8.314\,462\,618\ldots$	J mol^{-1} K^{-1}	exact
Faraday constant $N_A e$	F	$96\,485.332\,12\ldots$	C mol^{-1}	exact
Stefan-Boltzmann constant $(\pi^2/60)k^4/\hbar^3c^2$	σ	$5.670\,374\,419\ldots \times 10^{-8}$	W m^{-2} K^{-4}	exact
	Non-SI units accepted for use with the SI			
electron volt (e/C) J	eV	$1.602\,176\,634 \times 10^{-19}$	J	exact
(unified) atomic mass unit $\frac{1}{12}m(^{12}\mathrm{C})$	u	$1.660\,539\,066\,60(50) \times 10^{-27}$	kg	3.0×10^{-10}

* The energy of a photon with frequency ν expressed in unit Hz is $E = h\nu$ in J. Unitary time evolution of the state of this photon is given by $\exp(-iEt/\hbar)|\varphi\rangle$, where $|\varphi\rangle$ is the photon state at time $t = 0$ and time is expressed in unit s. The ratio $Et/\hbar$ is a phase.