14 March 2018, 1–5pm

Singapore Physics Olympiad Training

2018 selection test for Asian and International Physics Olympiad travelling teams

1. This is a **four-hour** test. Attempt all questions, the maximum total score is **82**. Marks allocated for each part of a question are indicated.
2. Check that there are a total of **10 printed pages** (including this cover page). The last page contains a table of physical constants that you may refer to.
3. Begin your answer for each question on a **fresh sheet of paper**, and present all answers clearly.
4. Write your name on the **top right hand corner of every answer sheet** you submit.
5. This cover sheet should be **stapled together** with your answer sheets, and the answer sheets should be properly sorted.
6. No books or documents relevant to the test may be brought into the examination room.
7. Please **complete and sign the declaration** on page 2.

Declaration

I declare that I will be fully committed to the training for and participation in the Asian Physics Olympiad and/or the International Physics Olympiad if selected for the travelling team. I will check first with the MOE coordinator before taking on additional commitments not listed below.

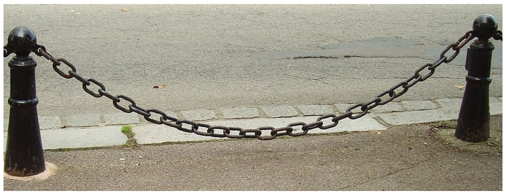
Potential limitations to my commitment in the period from now to end-July 2018 are described exhaustively in the box below, such as other academic competitions, CCA commitments (school-related or otherwise), travel plans, etc.

Name and signature: ______________________________

Question:	1	2	3	4	5	6	7	Total
Points:	28	9	12	14	11	4	4	82
Score:								

1. A chain has uniform linear mass density λ.

   (a) The chain is attached at its ends to two points that are at the same horizontal level, and it hangs freely (without touching the ground) in a uniform gravitational field of strength g. The distance between the two points of suspension is d. Let T_0 denote the tension in the chain at its lowest point.

   i. Qualitatively, what happens to the value of T_0 when d is reduced, assuming nothing else changes and the chain continues to hang freely? (No need to discuss the mathematical relationship between d and T_0.) [1]

   ii. Denote the horizontal and vertical axes as the x-axis and y-axis respectively (let the upwards direction be positive). Derive a second-order differential equation in $y = y(x)$ that the shape of the chain must obey. [5]

   iii. Show that [5]

   $$y = a \cosh\left(\frac{x}{a}\right) + b$$

   is a solution to the differential equation obtained, for an appropriate choice of origin for the axes. Relate the variables a and b to the parameters given earlier in the problem.

   (b) Through a siphoning mechanism, a sufficiently long chain can launch into a fountain. We consider the chain with uniform linear mass density λ initially on an elevated surface, with one end then allowed to fall onto the ground. This is well discussed in the paper by J. S. Biggins and M. Warner, *Proc. R. Soc. A* **470**, 20130689 (2014), from which this question is adapted.

   In this part of the problem, we start with a toy model of the phenomenon in the steady-state when the chain is moving with speed v. Label the tension in three regions as T_T, T_C, T_F with the table-to-floor height h_1 and the rise height of the fountain h_2 (see figure and caption taken from Biggins and Warner).

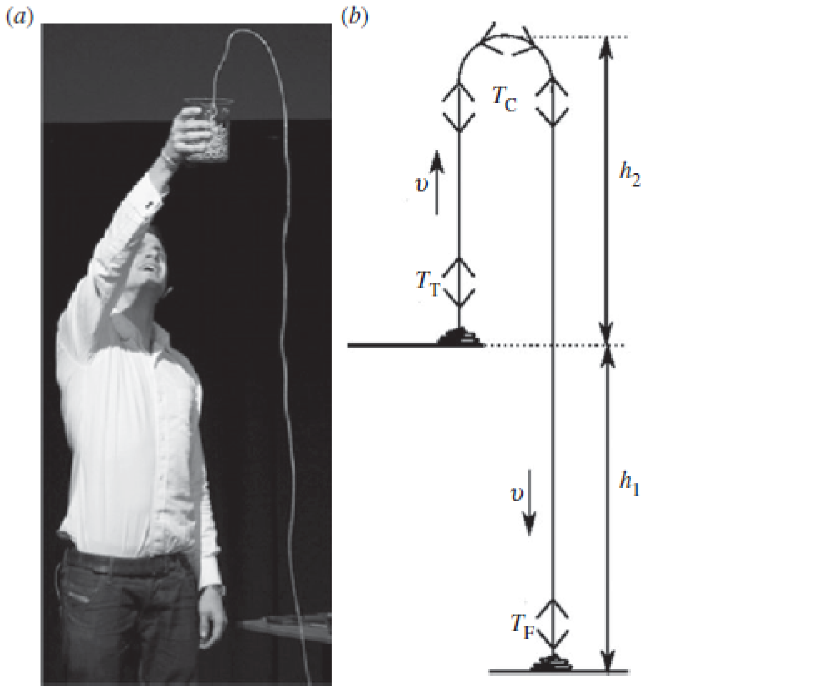


Figure 1. (*a*) Mould demonstrates a chain fountain. We thank J. Sanderson for permission to reproduce this photograph. (*b*) Our minimal model of a chain fountain. A chain, with mass per unit length λ, is in a pile on a flat table a distance h_1 above the floor. It flows to the floor at a speed v along the sketched trajectory, with T_T being the tension just above the table, T_C the tension in the small curved section at the top of the fountain and T_F the tension just above the floor.

i. Let r be the local radius of curvature for the section of the chain when it is at its highest point.

   α) By considering an infinitesimal section of the curved chain, derive an equation relating T_C with v and show that T_C is independent of r. You may assume that $v^2 \gg gr$. [2]

   β) Describe the physical interpretation of the assumption that $v^2 \gg gr$. [1]

ii. By considering the vertically rising and falling branches of the motion, derive two separate equations relating the various tensions. [2]

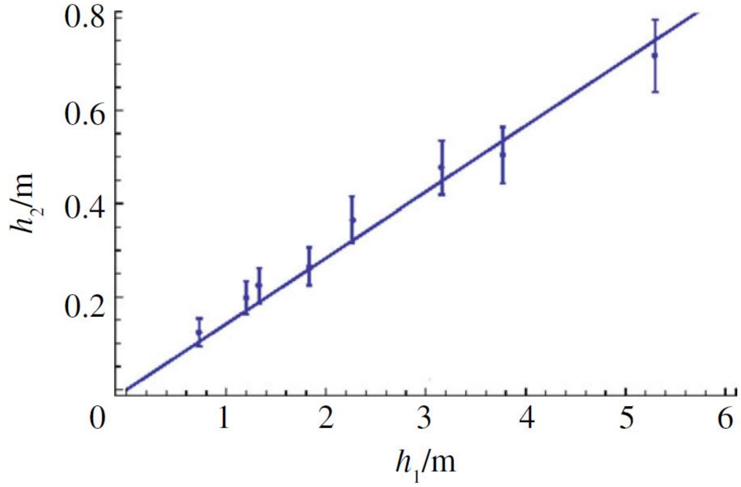
iii. By considering an infinitesimal section of the chain just as it rises from the table, derive an equation relating T_T with v. [1]

iv. Suppose also that $T_F = 0$ because the floor provides the full force required to bring the chain to rest. Discuss what you can deduce about h_2 and v for this model. [2]

(c) In this next part of the problem, we modify the model slightly to allow the table to exert a normal contact force $R = \alpha T_C$ on the chain as it lifts off. We also suppose $T_F = \beta T_C \geq 0$ such that the floor does not necessarily provide the full force that brings the chain to rest.

   i. Derive expressions for the ratios h_2/h_1 and v^2/gh_1 with this modification. [2]

   ii. From the experimental data below (taken from Biggins and Warner), suggest possible values for α, β. [2]



[Aside: Biggins and Warner conducted rather interesting numerical simulations using long chains of springs and masses, where other than simulating the physical setup, they had a version where the table was replaced by a region of zero gravity so that chain could "float" without the possibility of a contact force R.]

(d) In this final part of the problem, we explore the mathematical shape of the chain fountain (moving at steady-state speed v).

In the case of the hanging chain in (a), the variables x and $y = y(x)$ described the shape and the tension could also be found as a function $T = T(x)$.

In the case of the chain fountain, let us now parameterise using s, the distance (arc length) along the chain, so the tension is $T = T(s)$ and the shape can be described by the angle the chain makes with the vertical $\theta = \theta(s)$.

i. By considering forces parallel to the chain, show that [1]

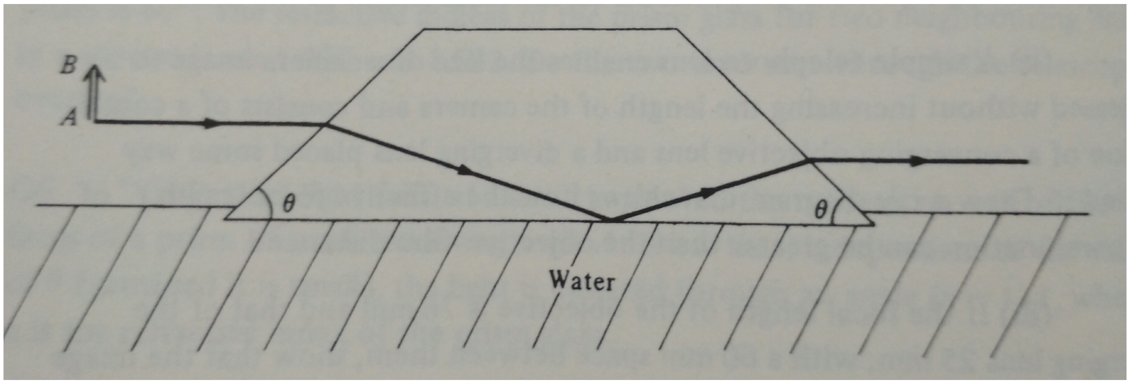
$$\frac{dT}{ds} = \lambda g \cos\theta \ .$$

ii. By considering forces perpendicular to the chain, derive an equation involving functions of $T(s)$ and $\theta(s)$ and their first derivatives. [2]
[**Hint:** the local curvature is $1/r = d\theta/ds$.]

iii. For the limiting case of $v = 0$, these equations describe the freely suspended chain and are solved by the hyperbolic cosine function given in the last part of (a). Discuss how the solutions for $\theta(s)$ and $T(s)$ are modified for $v \neq 0$. [2]
(Do not worry about boundary conditions. It's quite complicated and you can think about this more after the exam.)

Q1 total: 28

2. The cross-section of a glass prism is an isosceles triangle with its pointy top chopped off as shown, and has its base immersed in water. Let the refractive indices of glass and water be n_g and n_w respectively, with $n_g > n_w$.



(a) Find the constraints on the angle θ such that a ray of light parallel to the top and bottom surfaces of the prism is totally reflected at the base of the prism. [4]

(b) The prism is now lifted completely out of the water and dried.

i. Discuss if and how the constraint on θ found in the earlier part changes. [2]

ii. What is the orientation of the transmitted image of the arrow AB? [1]

iii. What is the orientation of the transmitted image of the arrow AB, when the prism is rotated an angle ϕ around the axis defined by the incident beam, for $\phi = \pi/2$ and $\phi = \pi$ respectively? [2]

Q2 total: 9

3. The van der Waals equation for a gas is given by

$$\left(p + a\frac{N^2}{V^2}\right)(V - bN) = Nk_BT$$

where p, V, T are the pressure, volume, and temperature of the gas respectively. N is the number of molecules of the gas, and k_B is Boltzmann's constant. The parameters a, b are allowed to be gas-dependent and assumed to be positive.

(a) Show that at constant temperature, the condition that dp/dV vanishes gives solutions that satisfy [4]

$$V - bN = \left(\frac{k_BT}{2Na}\right)^{1/2} V^{3/2} .$$

State an assumption made about the solutions.

(b) By sketching appropriate graphs, show that there are either zero, one, or two solutions to this equation. [2]

(c) Find expressions for $V = V_0$ and $T = T_0$ in the case that there is only one solution. [3]

(d) Sketch a few isothermal curves on a p-V diagram for temperatures below, at, and above T_0. [3]

Q3 total: 12

4. (a) A charged conducting ball has radius r and surface charge density σ.

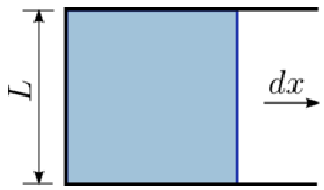
i. Find its electrostatic energy U. Explain your reasoning carefully. [3]

ii. By considering how U changes with r, assuming that the total charge on the ball is constant, derive an expression for the pressure p acting on the surface. [2]

(b) The tensile strength T of a material can be determined by fabricating a thin rectangular sheet with uniform thickness δ and a width L, then pulling on the sheet perpendicular to the width and along the direction of the sheet. T can then be defined as

$$T = \frac{F}{L\delta} \,,$$

where F is the minimum force needed to tear the sheet apart.



i. Show that, for a uniform spherical shell of radius r, the maximal pressure p^* is related to the tensile strength T by the formula [3]

$$p^* = \frac{2T\delta}{r} \,.$$

You should work from first principles and consider a small area on the surface of a sphere.

ii. A metal is chosen to fabricate a thin spherical shell of thickness $\delta = 1$ μm and radius $r = 0.5$ m to act as a capacitor. Find the minimum tensile strength T required such that the shell does not rupture before it discharges through the surrounding air. Assume that air breaks down at an electric field strength of $E_0 = 3$ MV/m. [3]

iii. Suppose the goal is to store a given amount of electrostatic energy, and we want to optimise for the amount of metal used. Is it better to use one large spherical shells or several smaller shells (keeping the shell thickness fixed in both cases)? Explain your answer clearly. [3]
[**Hint:** separately consider the two different physical mechanisms that impose constraints on energy storage.]

Q4 total: 14

5. This question explores the physics behind Planck's law for blackbody radiation. A blackbody is one that absorbs all radiation incident upon it. We model a blackbody by a photon gas in thermal equilibrium in a large cavity with a tiny hole. The radiation emitted from the hole will be characteristic of a perfect blackbody.

(a) Photon energy modes in a cavity.

i. Consider first a one-dimensional cavity, bounded by parallel (electrically) conducting plates separated by a distance L. Show that the allowed energy modes in the cavity are given by [2]

$$\varepsilon = \frac{nhc}{2L} \,,$$

where h is Planck's constant, c is the speed of light, and $n \in \mathbb{Z}^+$, the set of positive integers. Explain the physical arguments used to derive this expression.
(Note: the energy of a mode is the energy required for a single photon in that mode, measured relative to the vacuum energy.)

ii. Now consider the similar situation in a three-dimensional cavity, i.e. a cube of length L. Derive an expression for the allowed energy of each mode $\varepsilon = \varepsilon(\underline{n})$, where $\underline{n} = (n_1, n_2, n_3)$ is an ordered triple of positive integers that indexes the mode. [2]

(b) Occupation numbers for a given energy mode.

For each energy mode $\underline{n}$, the number of photons in the mode is a non-negative integer, which we denote as $r \in \mathbb{Z}^+ \cup \{0\}$. Each additional photon in the mode increases the energy by $\varepsilon(\underline{n})$, so according to statistical thermodynamics, the probability distribution for the number r of photons in the mode $\underline{n}$ is

$$P_{\underline{n}}(r) = \frac{1}{\mathcal{Z}_{\underline{n}}} e^{-r\beta\varepsilon(\underline{n})} ,$$

where $\beta \equiv 1/(k_B T)$ is the inverse temperature, and the normalisation factor

$$\mathcal{Z}_{\underline{n}} = \sum_{r=0}^{\infty} e^{-r\beta\varepsilon(\underline{n})} = \frac{1}{1 - e^{-\beta\varepsilon(\underline{n})}}$$

is known as the partition function for mode $\underline{n}$.

i. Show that the expectation value of r for mode $\underline{n}$ is given by [1]

$$\langle r \rangle_{\underline{n}} = -\frac{1}{\varepsilon(\underline{n})} \frac{d(\log \mathcal{Z}_{\underline{n}})}{d\beta} .$$

ii. Show that the total energy of photons in the cube is [2]

$$U = 2 \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \sum_{n_3=0}^{\infty} \frac{\varepsilon(\underline{n})}{e^{\beta\varepsilon(\underline{n})} - 1} ,$$

where the factor of two arises from the two independent modes for each $\underline{n}$ (related to the photon having spin 1).

(c) Large cavity limit.

In the limit of $L \to \infty$, the summation involves infinitesimal quantities and we have

$$U \propto \int_0^{\infty} \int_0^{\infty} \int_0^{\infty} d\epsilon_1\, d\epsilon_2\, d\epsilon_3\, \frac{\varepsilon(\underline{n})}{e^{\beta\varepsilon(\underline{n})} - 1} ,$$

where $\epsilon_1, \epsilon_2, \epsilon_3$ are some continuous variables with dimensions of energy.

i. Show that [2]

$$\frac{U}{L^3} = \int_0^{\infty} u_\nu \, d\nu ,$$

where $\nu = \epsilon/h$ is the frequency of a photon with energy ϵ.

ii. The spectral radiance of the radiation emitted from the hole in the cavity is $B_\nu = u_\nu \frac{c}{4\pi}$, where c is the speed of light. The expression you would have obtained (if done correctly!) for B_ν is known as Planck's law. [2]

Show that the expression you obtained matches the Rayleigh-Jeans law at low frequency

$$B_\nu = \frac{2}{c^2} \nu^2 k_B T$$

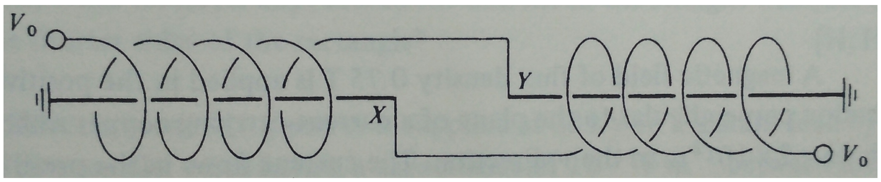
and the Wien approximation at high frequency

$$B_\nu = \frac{2h}{c^2} \nu^3 e^{-h\nu/(k_B T)} .$$

Q5 total: 11

6. The device shown schematically below consists of two identical interconnected units, with potential V_0 applied relative to the ground as shown.

   Each unit contains a long coil of resistance of $R = 2\ \Omega$, wound with $n = 10^4$ turns per metre, and along the axis of which runs a superconducting wire. Each superconducting wire (labelled X and Y) has zero resistance when the magnetic flux density is below $B_0 = 10^{-2}$ T, but has a resistance of $S = 10\ \Omega$ when driven into the normal (non-superconducting) state by a sufficiently large magnetic field.



   (a) Describe and explain how the device can act as a bistable logic element. [3]

   (b) Calculate the limits for V_0. [1]

Q6 total: 4

7. An interstellar physics teacher travels at constant velocity v relative to Earth. As her rocket grazes past her class of Earth-bound students, she sends out a signal for them to begin their test. The teacher would like the class to have time T to complete the test. [4]

   When should she send out a beam of light back to Earth in order to signal the end of the test?

Q7 total: 4

Physical constants

Speed of light in vacuum	c	=	$299\,792\,458\ \mathrm{m \cdot s^{-1}}$
Vacuum permeability (magnetic constant)	μ_0	=	$4\pi \times 10^{-7}\ \mathrm{kg \cdot m \cdot A^{-2} \cdot s^{-2}}$
Vacuum permittivity (electrical constant)	ε_0	=	$8.854\,187\,817 \times 10^{-12}\ \mathrm{A^2 \cdot s^4 \cdot kg^{-1} \cdot m^{-3}}$
Elementary charge	e	=	$1.602\,176\,620\,8(98) \times 10^{-19}\ \mathrm{A \cdot s}$
Mass of the electron	m_e	=	$9.109\,383\,56(11) \times 10^{-31}\ \mathrm{kg}$
		=	$0.510\,998\,946\,1(31)\ \frac{\mathrm{MeV}}{\mathrm{c^2}}$
Mass of the proton	m_p	=	$1.672\,621\,898(21) \times 10^{-27}\ \mathrm{kg}$
		=	$938.272\,081\,3(58)\ \frac{\mathrm{MeV}}{\mathrm{c^2}}$
Mass of the neutron	m_n	=	$1.674\,927\,471(21) \times 10^{-27}\ \mathrm{kg}$
		=	$939.565\,413\,3(58)\ \frac{\mathrm{MeV}}{\mathrm{c^2}}$
Unified atomic mass unit	u	=	$1.660\,539\,040(20) \times 10^{-27}\ \mathrm{kg}$
Rydberg constant	R_∞	=	$10\,973\,731.568\,508(65)\ \mathrm{m^{-1}}$
Universal constant of gravitation	G	=	$6.674\,08(31) \times 10^{-11}\ \mathrm{m^3 \cdot kg^{-1} \cdot s^{-2}}$
Acceleration due to gravity (in Zurich)	g	=	$9.81\ \mathrm{m \cdot s^{-2}}$
Planck's constant	h	=	$6.626\,070\,040\ (81) \times 10^{-34}\ \mathrm{kg \cdot m^2 \cdot s^{-1}}$
Avogadro number	N_A	=	$6.022\,140\,857\ (74) \times 10^{23}\ \mathrm{mol^{-1}}$
Molar gas constant	R	=	$8.314\,4598(48)\ \mathrm{kg \cdot m^2 \cdot s^{-2} \cdot mol^{-1} \cdot K^{-1}}$
Molar mass constant	M_u	=	$1 \times 10^{-3}\ \mathrm{kg \cdot mol^{-1}}$
Boltzmann constant	k_B	=	$1.380\,648\,52(79) \times 10^{-23}\ \mathrm{kg \cdot m^2 \cdot s^{-2} \cdot K^{-1}}$
Stefan-Boltzmann constant	σ	=	$5.670\,367\ (13) \times 10^{-8}\ \mathrm{kg \cdot s^{-3} \cdot K^{-4}}$