Singapore Physics Olympiad Team 2017

Selection test for Asian and International Physics Olympiad travelling teams

Instructions to Candidates:

1. This is a **four-hour** test.
2. Attempt all questions, the maximum total score is **80 marks**. Marks allocated for each part of a question are indicated in square brackets [ ].
3. Check that there are a total of **11 printed pages (including this cover page)**. Page 2 contains a table of physical constants that you may refer to.
4. Begin your answer for **each question** on a fresh sheet of paper, and present answers clearly.
5. Write your **name** on the top right hand corner of every answer sheet you submit.
6. Submit this question paper and all your working sheets. No paper, whether used or unused, may be taken out of this examination room.
7. No books or documents relevant to the test may be brought into the examination hall.
8. Please complete the declaration below.

Declaration:

I declare that I will be fully committed to the training for and participation in the Asian Physics Olympiad and/or the International Physics Olympiad if selected for the travelling team. Potential limitations to my commitment in the period from now to **mid-July 2017** are described <u>exhaustively</u> in the box below, e.g. other academic competitions, CCA commitments (school-related or otherwise), travel plans, etc.

Name: ______________________________________ Signature: _____________________

Potential limitations to my commitment are as follows:

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Table of Physical Constants

Speed of light in vacuum	c	=	$299\,792\,458\ \mathrm{m\cdot s^{-1}}$
Vacuum permeability (magnetic constant)	μ_0	=	$4\pi\times10^{-7}\ \mathrm{kg\cdot m\cdot A^{-2}\cdot s^{-2}}$
Vacuum permittivity (electrical constant)	ε_0	=	$8.854\,187\,817\times10^{-12}\ \mathrm{A^{2}\cdot s^{4}\cdot kg^{-1}\cdot m^{-3}}$
Elementary charge	e	=	$1.602\,176\,620\,8(98)\times10^{-19}\ \mathrm{A\cdot s}$
Mass of the electron	m_e	=	$9.109\,383\,56(11)\times10^{-31}\ \mathrm{kg}$
		=	$0.510\,998\,946\,1(31)\ \frac{\mathrm{MeV}}{\mathrm{c^2}}$
Mass of the proton	m_p	=	$1.672\,621\,898(21)\times10^{-27}\ \mathrm{kg}$
		=	$938.272\,081\,3(58)\ \frac{\mathrm{MeV}}{\mathrm{c^2}}$
Mass of the neutron	m_n	=	$1.674\,927\,471(21)\times10^{-27}\ \mathrm{kg}$
		=	$939.565\,413\,3(58)\ \frac{\mathrm{MeV}}{\mathrm{c^2}}$
Unified atomic mass unit	u	=	$1.660\,539\,040(20)\times10^{-27}\ \mathrm{kg}$
Rydberg constant	R_∞	=	$10\,973\,731.568\,508(65)\ \mathrm{m^{-1}}$
Universal constant of gravitation	G	=	$6.674\,08(31)\times10^{-11}\ \mathrm{m^{3}\cdot kg^{-1}\cdot s^{-2}}$
Acceleration due to gravity (in Zurich)	g	=	$9.81\ \mathrm{m\cdot s^{-2}}$
Planck's constant	h	=	$6.626\,070\,040\,(81)\times10^{-34}\ \mathrm{kg\cdot m^{2}\cdot s^{-1}}$
Avogadro number	N_A	=	$6.022\,140\,857\,(74)\times10^{23}\ \mathrm{mol^{-1}}$
Molar gas constant	R	=	$8.314\,4598(48)\ \mathrm{kg\cdot m^{2}\cdot s^{-2}\cdot mol^{-1}\cdot K^{-1}}$
Molar mass constant	M_u	=	$1\times10^{-3}\ \mathrm{kg\cdot mol^{-1}}$
Boltzmann constant	k_B	=	$1.380\,648\,52(79)\times10^{-23}\ \mathrm{kg\cdot m^{2}\cdot s^{-2}\cdot K^{-1}}$
Stefan-Boltzmann constant	σ	=	$5.670\,367\,(13)\times10^{-8}\ \mathrm{kg\cdot s^{-3}\cdot K^{-4}}$

Question 1 – Billiards [5]

A billiard ball, a hard sphere of uniform density, has mass *m* and radius *r*. It rolls without slipping on a felt-lined billiard table. The ball moves directly towards the edge of the table, and hits the overhang (see **Fig 1.1** and **Fig 1.2**).

Determine the height *h* of the overhang, such that the ball rolls without slipping immediately after the collision. State and explain any additional assumptions required.

{You may wish to note that the moment of inertia for a thin spherical shell of mass *M* and radius *R* about an axis through its centre is $\frac{2}{3}MR^2$.}

[5 marks]

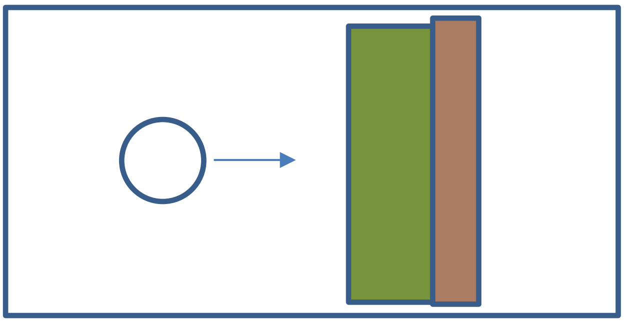

Figure 1.1: Top view of billiard ball rolling towards the edge of the table

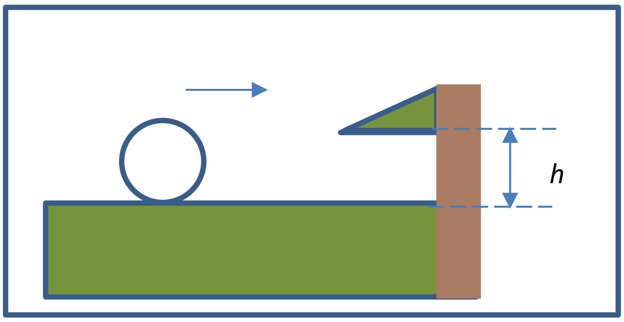

Figure 1.2: Side view of billiard ball rolling towards the edge of the table

Question 2 – Oscillations [15]

A point object, mass *m*, slides on a frictionless parabolic curve, in a uniform gravitational field *g*. The curve shown in **Fig 2.1** has the equation $y = \frac{x^2}{L}$, where *L* is a constant.

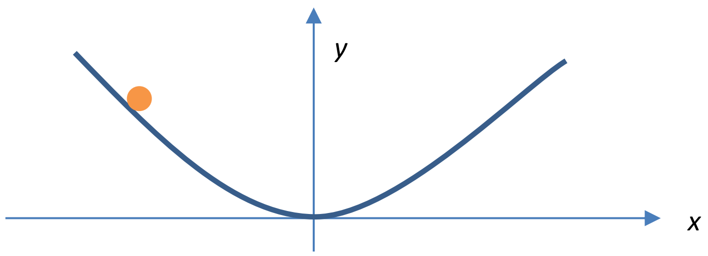

Figure 2.1: Point object sliding on a frictionless parabolic curve

(a)(i) Derive an expression for the total energy $E_{parabolic}$ of the point object, as a function only of *x* and $\dot{x}$, the time derivative of *x*.

[3 marks]

(a)(ii) Show that the motion of the point object is only **approximately** simple harmonic, and find the period T_{approx}.

[3 marks]

Instead of the frictionless parabolic curve, we want to find the frictionless tautochrone[1] curve, on which a point object will oscillate with the same period regardless of where it is released from (i.e. period independent of amplitude, just like for a simple harmonic oscillator).

Let this period of oscillation be T_0.

(b)(i) Define the arc length *s* as a variable that gives the distance along the curve from the origin. Derive an expression for the total energy $E_{tautochrone}$ of the point object, written as a function only of *s* and $\dot{s}$, the time derivative of *s*. {Hint: If you get stuck, some dimensional analysis might be helpful and could even gain you some marks!}

[4 marks]

(b)(ii) Deduce the parametric equations defining the tautochrone curve, i.e. the expressions $x(\theta), y(\theta)$ where the angle θ is related to the slope of the curve, i.e. $\tan\theta = \frac{dy}{dx}$.

[5 marks]

[1] The Greek *tauto-* means "same" (e.g., tautology), while *chrone* means "time" (e.g., chronology).

Question 3 – Soap Bubble [5]

A bubble machine produces a spherical soap bubble of 5 cm diameter and thickness 0.5 µm, and increases the electric potential of the bubble by +10 V as it is being produced. The bubble floats around a bit and collapses into a drop.

Estimate the electric potential of the drop. State and explain any additional assumptions required.

[5 marks]

Question 4 – Balance [5]

Determine the necessary and sufficient conditions for the detector in the circuit diagram **Fig 4.1** to pick up zero signal, for an alternating voltage with peak amplitude V_0 and angular frequency ω. Comment on your answer.

[5 marks]

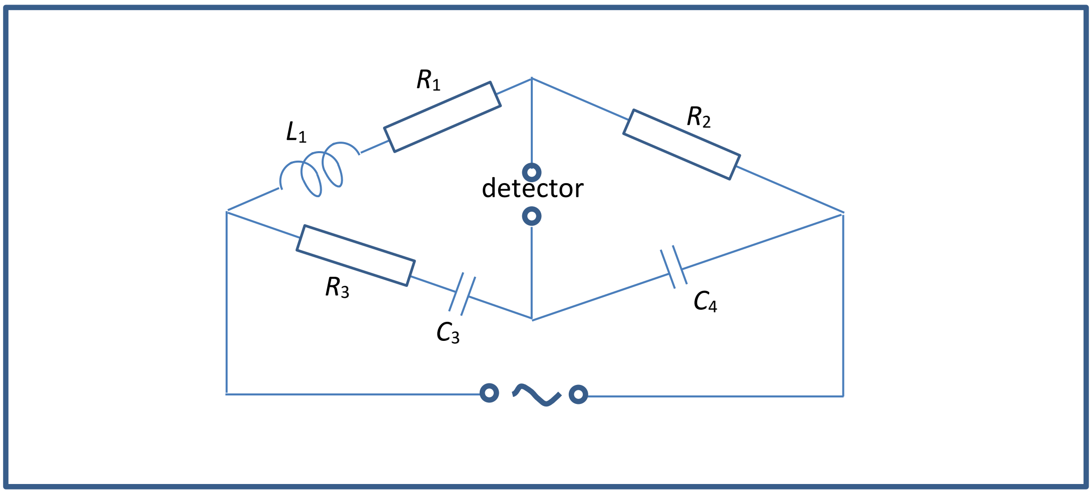

Figure 4.1: Circuit diagram with detector

Question 5 – Relativistic Doppler [10]

Frame S' moves with velocity v relative to the inertial frame S, as shown in **Fig 5.1**. The coordinates in S and S' are (x, t) and (x', t') respectively. The velocity v is in the positive x-direction, and is less than the speed of light c.

At $t = t' = 0$, we have $x = x' = 0$, such that the origins O and O' of the coordinate frames coincide.

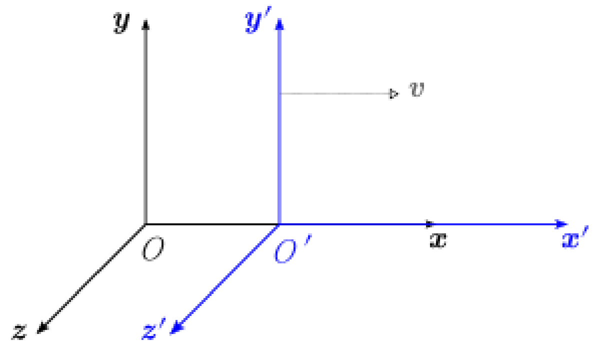

Figure 5.1: Inertial frames of reference S and S'

There is a wave source fixed at the origin O of the frame S. The source produces waves of frequency f and speed u, as measured by Alice, an observer (at rest) in frame S.

Bob is an observer (at rest) in frame S'.

Alice notices that the wave source produces a pulse at $(x_1, t_1) = (0, 0)$.

(a) What are the coordinates (x_1', t_1') of this event according to Bob?

[1 mark]

Alice notices the next pulse from the source at $(x_2, t_2) = (0, T)$.

(b)(i) Write down an expression for T.

[1 mark]

(b)(ii) What are the coordinates (x_2', t_2') of this event according to Bob?

[2 marks]

For Bob to judge the frequency of the wave source from his frame of reference, he can calculate the time interval between the two pulses.

(c)(i) What is the speed u' of the wave according to Bob?

[1 mark]

(c)(ii) Show that the frequency as determined by Bob is

$$f' = f \cdot \left(1 - \frac{v}{u}\right) \cdot \left(1 - \frac{v^2}{c^2}\right)^{-1/2} \qquad \textit{(Equation 5.1)}$$

[3 marks]

(c)(iii) Explain how to obtain, from *Eqn 5.1*, the expression for the relativistic Doppler shift for electromagnetic waves.

[1 mark]

(c)(iv) What does the $c \to \infty$ limit of *Eqn 5.1* correspond to?

[1 mark]

Question 6 – Superfluid [10]

In a superfluid phase, particles move with zero viscosity. In this question, we explore a theoretical model, due to Lev Landau (1962 Nobel Prize in Physics), for how drag forces can be absent in a superfluid. The basic insight is that if the particles do **not** interact, then there is no dissipation or drag.

We describe an analogy, to help make clearer[2] what you are asked to imagine. A neutron travelling through the quantum vacuum might "interact" with the vacuum to decay into a proton and an electron (and a neutrino). For a microscopic particle travelling in a fluid, assume that the interaction of the particle with the fluid produces a quasiparticle[3], which you should treat as a regular particle.

Let the microscopic particle travelling in the fluid have mass *M*, and velocity $\underline{\boldsymbol{u}}$ and $\underline{\boldsymbol{v}}$ respectively before and after the interaction with the fluid. Let the energy and momentum of the quasiparticle produced be denoted by ε and $\underline{\boldsymbol{p}}$ respectively. Note that the underlined quantities indicate vectors.

(a) Write down equation(s) relating the various quantities introduced above, and justify their validity.

[2 marks]

(b) Show that the equation(s) in part (a) can only be satisfied if

$$u > \min_{p} \frac{\varepsilon(p)}{p}$$

where $u = |\underline{\boldsymbol{u}}|$ and $p = |\underline{\boldsymbol{p}}|$.

[3 marks]

[2] Hopefully this doesn't end up confusing you more!
[3] The term quasiparticle is used to denote that the "particle" produced is the result of a collective excitation, similar to how phonons are the result of lattice vibrations in a solid.

(c)(i) By considering that the microscopic particle could in fact be a fluid particle, discuss how to obtain a "superfluid criterion" to decide whether a system is in the superfluid phase or not, based on the quasiparticle dispersion $\varepsilon(p)$.

[1 mark]

(c)(ii) Apply this criterion to a system where the quasiparticles of mass *m* obey:

1. the free particle spectrum $\varepsilon = \frac{p^2}{2m}$

[1 mark]

2. the so-called Bogoliubov spectrum $\varepsilon = p\sqrt{c^2 + \frac{p^2}{4m^2}}$

[1 mark]

3. the spectrum for liquid helium-4 at a temperature of 2.0 K, shown in **Fig 6.1**

[2 marks]

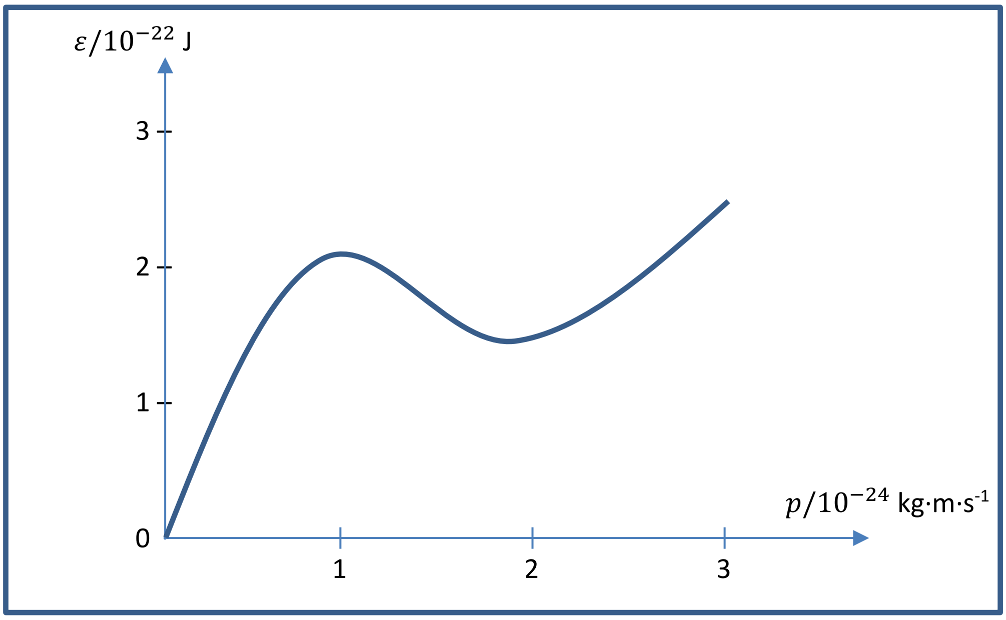

Figure 6.1: Quasiparticle dispersion for helium-4 at 2.0 K

Question 7 – Cooling [20]

Consider a monatomic ideal gas in two spatial dimensions, constrained in a "box" of area *A*, as shown in **Fig 7.1**. Each atom has mass *m*. The number of atoms in the gas is *N*, and the temperature of the gas is *T*.

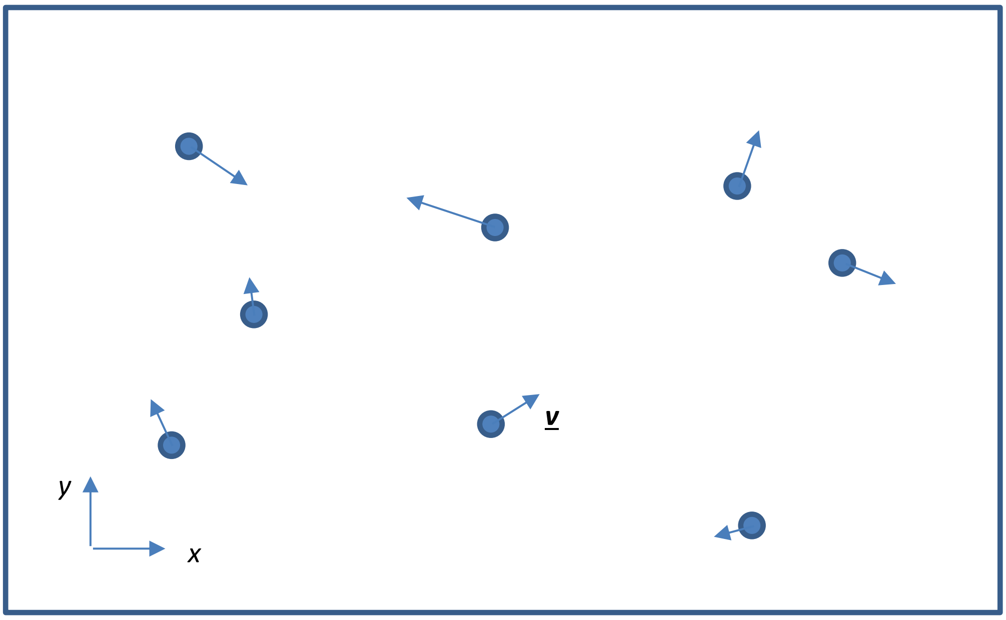

Figure 7.1: Two dimensional monatomic ideal gas

(a)(i) Explain clearly why the infinitesimal probability $d^2P(v_x, v_y)$ for an atom to have the velocity in the range $([v_x, v_x + dv_x], [v_y, v_y + dv_y])$ is given by

$$d^2P(v_x, v_y) = C \exp\left(-\frac{m(v_x^2 + v_y^2)}{2k_BT}\right) dv_x dv_y$$

[2 marks]

(a)(ii) And determine an expression for *C*. {Hint: You may use the identity $\int_{-\infty}^{\infty} e^{-x^2}\, dx = \sqrt{\pi}$.}

[2 marks]

(a)(iii) Deduce the expression for the infinitesimal probability dP_v for an atom to have a speed between v and $(v + dv)$.

[2 marks]

It is experimentally possible to rapidly (within a time-interval τ) remove all particles with speed $v > \sqrt{\frac{2\eta k_BT}{m}}$, without affecting the other particles.

(b)(i) Derive an expression for the number of atoms N' left in the box after this process, as a function of the parameter η.

[2 marks]

(b)(ii) Sketch a graph to show the variation of $\frac{N'}{N}$ against η.

[1 mark]

(b)(iii) Derive an expression for the total energy of the gas left in the box, in terms of m, N, T, η.

[2 marks]

The gas will re-thermalise after a time $t \gg \tau$, and reach a new equilibrium temperature T'.

(c)(i) Explain qualitatively why cooling occurs.

[1 mark]

(c)(ii) Derive an expression for T', and show that $T' \leq T$.

[3 marks]

(c)(iii) Sketch a graph to show the variation of $\frac{T'}{T}$ against η.

[1 marks]

In the framework of quantum statistical mechanics, we define the phase space density

$$\rho = \frac{N\lambda^2}{A}$$

where $\lambda = \frac{h}{\sqrt{2\pi m k_B T}}$ is the thermal de Broglie wavelength.

(d)(i) Derive an expression for ρ'/ρ, the ratio of phase space density after the cooling process to the phase space density before cooling.

[2 marks]

(d)(ii) Sketch a graph to show the variation of $\frac{\rho'}{\rho}$ against η.

[1 mark]

(d)(iii) Suggest the implications of phase space density on such a cooling scheme.

[1 mark]

Question 8 – Time-of-flight Thermometry [10]

A cloud of rubidium-87 atoms, with atomic mass m = 1.45 x 10^{-25} kg, is confined in a monochromatic laser trap with angular frequency ω. The temperature of the gas is T.

(a) On dimensional grounds, suggest an expression for the mean square radius $\langle R^2 \rangle$ of the atomic cloud.

[1 mark]

The trapping lasers are abruptly switched off at time $t = 0$.

(b)(i) Explain why $\langle R^2 \rangle(t)$ is expected to evolve in a spatially isotropic fashion, even in the presence of gravity in the vertical direction.

[1 mark]

(b)(ii) On dimensional grounds, suggest an expression for the time-evolution of the mean squared radius of the cloud, $\langle R^2 \rangle(t) - \langle R^2 \rangle(0)$.

[1 mark]

The results of such an experiment are shown in **Tab 8.1**.

t / ms	2	3	4	5	6	7	8	9
$\sqrt{\langle R^2 \rangle}$ / µm	464	528	596	670	785	869	1010	1105

Table 8.1: Experimental data from time-of-flight measurements on a rubidium cloud

(c) Plot an appropriate graph and determine the temperature T.

[7 marks]