Thirtieth Singapore Physics Olympiad Theoretical Paper Part B

Saturday 21 October 2017

Organised by
Institute of Physics

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Instructions to Candidates

1. This is a 1.5 hour paper.
2. This paper consists of three (3) questions printed on sixteen (16) pages.
3. Attempt all questions. Marks allocated for each part of a question are indicated in the brackets [ ].
4. Write your answers in the space provided in the booklet.
5. If you need working paper, you may request from the invigilators.
6. No books or documents relevant to the test may be referred during the examination.

NAME: ____

SCHOOL: ____

5a. [5 marks] Consider two inertial frames S and S' whereby observer is stationary in S. S' frame is moving with relativistic speed v in the in x-direction with respect to the observer. The relation between the spacetime coordinates of S and S' frame is given by the Lorentz transformation

$$\begin{aligned} ct' &= \gamma\,(ct - \beta x) \\ x' &= \gamma\,(x - \beta ct) \\ y' &= y \\ z' &= z \end{aligned}$$

where

$$\gamma = \frac{1}{\sqrt{1-\beta^2}}$$

and

$$\beta = \frac{v}{c}.$$

Suppose the quantity $\mathbf{k}\cdot\mathbf{r} - \omega t$ is an invariant quantity, that is

$$\mathbf{k}'\cdot\mathbf{r}' - \omega' t' = \mathbf{k}\cdot\mathbf{r} - \omega t$$

where $k \equiv |\mathbf{k}| = \omega/c$ is the wave vector, $\mathbf{r}$ is the position vector and ω is the angular frequency.

Find ω', $c\mathbf{k}'$ in terms of ω, $\mathbf{k}$, and γ.

(b) [5 marks] A plane mirror (S' frame) is moving through vacuum with relativistic speed v in the x direction. As seen by observer in S, a beam of light with frequency ω_i is incident (from $x = +\infty$), at angle θ_i to the normal, on the plane mirror. The beam of light is reflected back at an angle θ_r to the normal and at frequency ω_r as shown in figure below. Here $\mathbf{k}_i$ and $\mathbf{k}_r$ are the incident and reflected wave vectors respectively.

Hint: Law of reflection is only valid in mirror's frame.

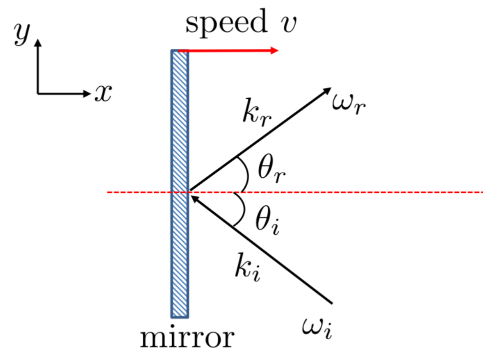

Figure 1: Reflection on a moving mirror

(i) What is the frequency of the reflected light ω_r expressed in terms of ω_i, β and θ_i ?

(ii) What is the energy of the each photon in the reflected beam?

6. Funny Balloon

The two balloons experiment is a famous counter-intuitive experiment involving two interconnected balloons of different radii. In this experiment, two identical balloons are initially inflated to different radii. They are then connected to both ends of a tube, allowing air flow between them. This results in the smaller balloon becoming smaller while the larger balloon becoming larger. This observation is surprising since conventional intuition suggests that both balloons should end up equal in sizes after allowing air flow between them. This question studies the physics behind this surprising result. For simplicity, we assume that the balloons are made up of spherical shells and all changes in sizes take place under isothermal conditions.

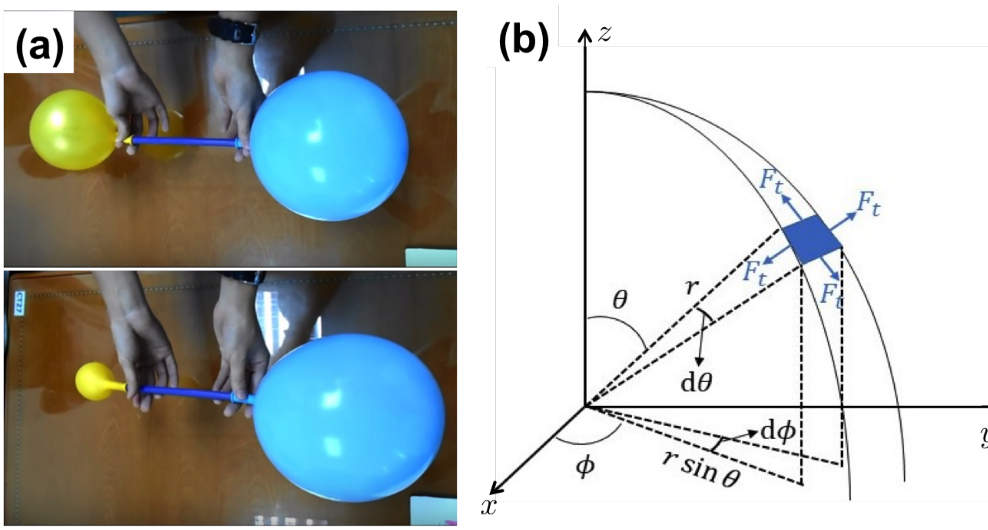

Figure 2: (a) The sizes of the balloons before and after exchanging air. (b) Tension in a rubber fragment making up a spherical rubber shell of radius r. The rubber fragment can be thought as an area formed by two arcs of two different circles. One edge of the rubber fragment is a segment from a circle at an angle ϕ from the $x-z$ plane, with radius r. The other edge is a segment from a circle in the $x-y$ plane, with radius $r\sin\theta$. There are tensions acting on all four sides of the rubber fragment. Each tension is in a direction tangent to the arc forming the fragment, pointing away from the fragment.

(a) Pressure-radius curve

The tension in a stretched rubber is given as

$$F_r \propto \frac{r}{r_0^2}\left[1-\left(\frac{r_0}{r}\right)^6\right]$$

where is r the radius of the inflated balloon and r_0 is the original radius of the balloon. Fig. 2(b) shows a sketch of the direction of tension in a rubber fragment that is part of the spherical shell.

(i) [3 marks] Show that the internal air pressure of the balloon relative to the atmospheric pressure is given by

$$P = \frac{C}{r_0^2 r}\left[1-\left(\frac{r_0}{r}\right)^6\right]$$

where C is a constant.

(ii) [1 mark] Show that the internal pressure of the balloon is maximum when the radius is

$$r_{max} = 7^{\frac{1}{6}} r_0$$

(iii) [2 marks] Sketch the pressure-radius graph of the balloon. Label the points r_0 and r_{max} on the graph.

6b. Suppose two balloons of two different radii are connected to two ends of a very thin tube with a valve, initially closed to prevent exchange of air between the balloons. The initial radius of the smaller balloon r_{si} is at its maximum as given

$$r_{si} = 7^{\frac{1}{6}} r_0$$

The balloons reach equilibrium after the valve is opened for some time. At equilibrium, the radius of the smaller balloon is

$$r_{sf} = 6^{\frac{1}{6}} r_0$$

(i) [2 marks] Calculate the radius of the bigger balloon r_{bf} at equilibrium. Leave your answer in 3 decimal places in terms of r_0. You may use numerical methods to solve this question.

(ii) [3 marks] Calculate the initial radius of the bigger balloon, r_{bi} when the valve was still closed. Leave your answer in 3 decimal places in terms of r_0. You may use numerical methods to solve this question.

(iii) [4 marks] Given the following quantities:

A = cross-sectional area of the tube,

m = average mass of one air molecule,

k = Boltzmann constant,

T = temperature at which the two balloons experiment is conducted,

r_b = radius of the bigger balloon at a given instant and

r_s = radius of the smaller balloon at a given instant,

show that the time taken for the balloons to reach the equilibrium state is given by

$$t = \frac{16\pi}{3A}\sqrt{\frac{m}{kT}}\left[\int_{r_{bi}}^{r_{bf}} dr_b \frac{r_b\left[1 - 2\left(\frac{r_0}{r_b}\right)^6\right]}{\frac{1}{r_s}\left[1 - \left(\frac{r_0}{r_s}\right)^6\right] - \frac{1}{r_b}\left[1 - \left(\frac{r_0}{r_b}\right)^6\right]}\right]$$

7a. [4 marks] Consider a volume V with bounding surface S. Suppose there is some electrostatic field $\mathbf{E}_{inside}(\mathbf{r})$ inside V and some other electrostatic field $\mathbf{E}_{outside}(\mathbf{r})$ outside V, show that at any point $\mathbf{r}$ on S we have

$$E^{\perp}_{outside}(\mathbf{r}) - E^{\perp}_{inside}(\mathbf{r}) = \frac{1}{\epsilon_0}\sigma(\mathbf{r}),$$

where $E^{\perp}(\mathbf{r})$ is the component of $\mathbf{E}(\mathbf{r})$ perpendicular to S at $\mathbf{r}$, $\sigma(\mathbf{r})$ is the charge per unit area at $\mathbf{r}$, and ϵ_0 is the electric permittivity of free space.

7b. [5 marks] Next suppose V is spherical and there is a continuous distribution of electric charge in V. If the charge per unit volume ρ is a constant, show that the charge per unit area on the surface S, which bounds V, is everywhere zero.

7c. [6 marks] Now suppose V is a spherical conductor of radius R carrying charge Q. If V is the only object in the "universe", calculate the total electric force experienced by the right half of V.

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General Data Sheet

Speed of light in vacuum	c	$=$	$299\,792\,458\ \mathrm{m \cdot s^{-1}}$
Vacuum permeability (magnetic constant)	μ_0	$=$	$4\pi \times 10^{-7}\ \mathrm{kg \cdot m \cdot A^{-2} \cdot s^{-2}}$
Vacuum permittivity (electrical constant)	ε_0	$=$	$8.854\,187\,817 \times 10^{-12}\ \mathrm{A^2 \cdot s^4 \cdot kg^{-1} \cdot m^{-3}}$
Elementary charge	e	$=$	$1.602\,176\,620\,8(98) \times 10^{-19}\ \mathrm{A \cdot s}$
Mass of the electron	m_e	$=$	$9.109\,383\,56(11) \times 10^{-31}\ \mathrm{kg}$
		$=$	$0.510\,998\,946\,1(31)\ \frac{\mathrm{MeV}}{\mathrm{c^2}}$
Mass of the proton	m_p	$=$	$1.672\,621\,898(21) \times 10^{-27}\ \mathrm{kg}$
		$=$	$938.272\,081\,3(58)\ \frac{\mathrm{MeV}}{\mathrm{c^2}}$
Mass of the neutron	m_n	$=$	$1.674\,927\,471(21) \times 10^{-27}\ \mathrm{kg}$
		$=$	$939.565\,413\,3(58)\ \frac{\mathrm{MeV}}{\mathrm{c^2}}$
Unified atomic mass unit	u	$=$	$1.660\,539\,040(20) \times 10^{-27}\ \mathrm{kg}$
Rydberg constant	R_∞	$=$	$10\,973\,731.568\,508(65)\ \mathrm{m^{-1}}$
Universal constant of gravitation	G	$=$	$6.674\,08(31) \times 10^{-11}\ \mathrm{m^3 \cdot kg^{-1} \cdot s^{-2}}$
Acceleration due to gravity (in Zurich)	g	$=$	$9.81\ \mathrm{m \cdot s^{-2}}$
Planck's constant	h	$=$	$6.626\,070\,040\,(81) \times 10^{-34}\ \mathrm{kg \cdot m^2 \cdot s^{-1}}$
Avogadro number	N_A	$=$	$6.022\,140\,857\,(74) \times 10^{23}\ \mathrm{mol^{-1}}$
Molar gas constant	R	$=$	$8.314\,4598(48)\ \mathrm{kg \cdot m^2 \cdot s^{-2} \cdot mol^{-1} \cdot K^{-1}}$
Molar mass constant	M_u	$=$	$1 \times 10^{-3}\ \mathrm{kg \cdot mol^{-1}}$
Boltzmann constant	k_B	$=$	$1.380\,648\,52(79) \times 10^{-23}\ \mathrm{kg \cdot m^2 \cdot s^{-2} \cdot K^{-1}}$
Stefan-Boltzmann constant	σ	$=$	$5.670\,367\,(13) \times 10^{-8}\ \mathrm{kg \cdot s^{-3} \cdot K^{-4}}$