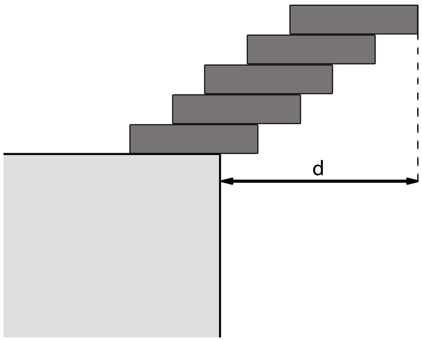
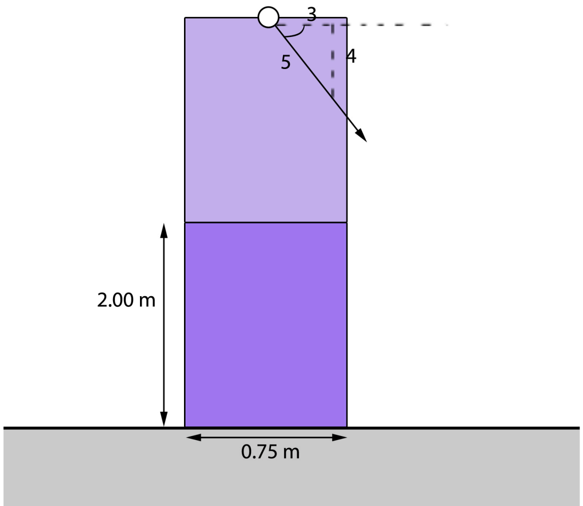
SPhO 2007 Theory Round (4 hours)

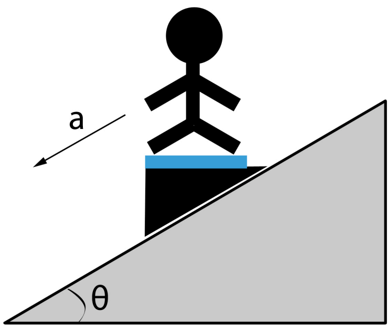
1. You have 5 blocks of dimension 1 meter by 1 meter and 25 mm thick. What is the furthest you can stack them from the edge of a table before they topple? [8 marks]



2. A column is formed of two marble blocks each weighing 15 kN sitting atop each other. A rope to which a force F is applied is attached at the top (of the top marble block). The friction coefficient between the marble and the ground is $\mu_{M-G} = 0.15$ while the friction coefficient between marble and marble is $u_{M-M} = 0.20$. What is the maximum load F_{max} that can be applied? [8 marks]

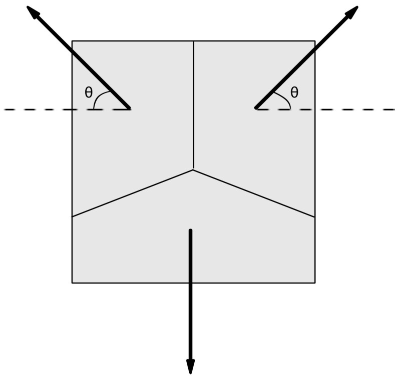


3. A weighing scale is fixed firmly to a wedge. A person of mass M is standing on the scale. The wedge is on a frictionless inclined ramp that makes an angle to the horizontal as shown in the figure. The system is sliding down the ramp with an acceleration a. The reading of the scale registers the apparent weight of the person. Derive an expression for the apparent weight W of the person as he slides down.



4. (a)A projectile is fired from ground level with an initial velocity of V_o at an angle of θ above the horizontal. Show that the maximum height H is given by $H = \dfrac{V_o^2 \sin^2\theta}{2g}$ where g is the acceleration of free fall. [4 marks]

(b) A rocket is projected upwards and explodes into three equally massive fragments just as it reaches the top of its flight. One of the fragments if observed to fall downwards in time t_1 as shown in Figure 4 while the other two land after a time t_2 after the burst. Determine the height $h(t_1, t_2)$ at which the fragmentation occurs. [8 marks]



5. Explain the following:
   (a) A feather and a piece of metal of roughly the same size and shape fall with equal acceleration on the Moon but not the Earth. [3 marks]

   (b) Forces between the molecules of a gas cause deviations from ideal gas behavior but forces between the walls of the container and the molecules do not. [3 marks]

   (c) The sun appears elliptical, as it is about to set over a distant horizon. [3 marks]

   (d) An electric motor in which the armature and field windings are in series slows down more when the load is increased than a motor in which they are in parallel. [3 marks]

6. A circular coil has inner radius R_1 and an outer radius R_2. The length of the coil is L.

   (i) Show that the magnetic induction B at the center when the coil carries a current I is

$$B = \frac{\mu_o n I L}{2} \ln\left[\frac{a + \sqrt{a^2 + b^2}}{1 + \sqrt{1 + b^2}}\right]$$

where $a = R_2/R_1$, $b = L/2R_1$ and n is the number of turns per square meter. [10 marks]

   (ii) Show that the length of the wire is $2\pi n(a^2 - 1)bR_1^3$.

7. A ship in a calm port nearing a shore radio station received a 200 MHz signal from the station's antenna. Both station's antenna and the ship's receiving antenna are located 20 m above the sea surface. A succession of maxima and minima are heard in the signal received at the ship. How far is the ship from the station the first time the signal passes through a minimum? How fast is the ship moving if the time between the first minimum and the next one is 50 s? [10 marks]

8. To account for the finite size of molecules and the attractive forces between molecules, Van der Waal devised an equation to describe the gas law for "real" gases $\left(P+\frac{a}{V^2}\right)(V-b)=RT$ where P, V and T are the pressure, volume and temperature of the gas; and R is the universal gas constant.

   (a) What are the dimensions of the constants a and b? [4 marks]

   (b) Show that the critical point temperature and pressure are given by

   $T_c=\frac{8a}{27Rb}$ and $P_c=\frac{a}{27b^2}$ [7 marks]

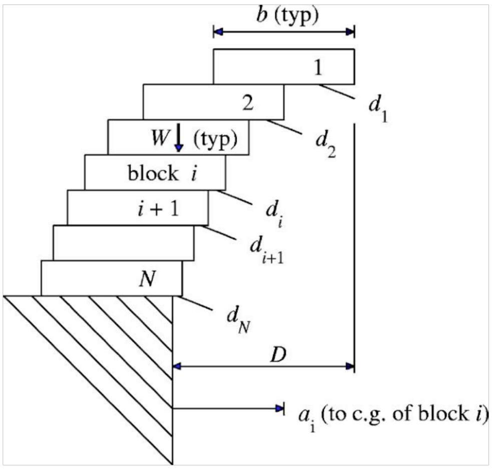
9. (a) Consider an oscillating element of air of cross sectional area A and thickness Δx whose centre is displaced from its equilibrium position by a distance x. We can write the pressure variation of the displaced element as $\Delta p=-B\frac{\Delta V}{V}$ where $V=A\Delta x$ and B is the bulk modulus. The volume change as a result unequal displacements of the two faces of the element differing by an amount Δs. By applying Newton's law of motion to the element, show that $\rho v^2=B$ where v is the speed of the element and ρ is the density of the element. [4 marks]

   (b) A period of pulsating star may be modeled as a star whose radius R varies periodically with time, performing radial longitudinal pulsations in the fundamental standing wave mode. By analogy with a pipe with one open end, determine the period of the pulsation T in terms of the radius and the average speed of sound in the star matter. For white dwarf stars composed of a material with a bulk modulus of 1.33 x 10^{22} Pa and a density of 10^{10} kg/m^3 and a radius of 9.0 x 10^{-3} solar radius, what is the approximate pulsation period of this star? [6 marks]

10. A photon of energy E is scattered from a free stationary electron of rest mass M_o. Show that the maximum kinetic energy of the recoiling electron is given by $K_{\max}=\frac{2E^2}{2E+M_oc^2}$.

Suggested Solutions

1)



In order for the system to remain stable, three conditions have to be fulfilled:

(1) The centre of mass of the whole system (CM_{sys}) of blocks must be before or on the edge of the table.

(2) The centre of mass of each block must be before or on the edge of the previous block.

(3) The centre of mass of any sub-system (ie. the blocks above a certain i^{th} block), must be before the edge of the i^{th} block.

As such, we will now consider the position of the centre of mass of the entire system.

Torque has to be balanced, and as such, for N number of blocks, the torque at block i is,

$$\frac{1}{i}\left(a_1 + a_2 + \cdots a_i\right) = a_{i+1} + \frac{l}{2} - \Delta_i, \qquad 1 \le i \le N\text{-}1$$

$$\frac{1}{N}\left(a_1 + a_2 + \cdots a_N\right) = -\Delta_N$$

where Δ_i is an arbitrary non-negative distance distance.

Solving for a_1 by resolving the above equations, we get

$$a_1 = C \cdot l - \Delta_N - \sum_{i=1}^{N-1} \frac{1}{i+1}\Delta_i$$

With C being a positive constant (we're not interested in).

To maximize a_1, for all values of i, $\Delta_i = 0$

Solving for a_i, $a_i = a_{i+1} + \dfrac{l}{2i}$

$$a_N = \frac{l}{2N} - \frac{l}{2}$$

Thus each d (as in figure) can be found as $d_N = a_i - a_{i+1} = \dfrac{l}{2N}$

Total distance $= \displaystyle\sum_{i=1}^{N} \frac{l}{2i} = \sum_{i=1}^{5} \sqrt{2}\,\frac{1}{2i} = \frac{77}{60}\sqrt{2} = 1.82m(3s.f.)$

2)

Force analysis will yield a few equations,

$f_1 =$ frictional force between the blocks $= \mu_{M-M}\left(W + F_{\max}\sin\theta\right)$

$f_2 =$ frictional force between the block and ground $= \mu_{M-G}\left(2W + F_{\max}\sin\theta\right)$.

To prevent slipping, $f_1 = f_2 = F_{\max}\cos\theta$

$\mu_{M-M}\left(W + F_{\max}\sin\theta\right) = \mu_{M-G}\left(2W + F_{\max}\sin\theta\right) = F_{\max}\cos\theta$

Analysis (testing for all three equations) will show that the equation $f_1 = F_{\max}\cos\theta$ is the limiting equation.

Substituting $\sin\theta = \frac{4}{5}, \cos\theta = \frac{3}{5}, \mu_{M-G} = 0.15, W = 15kN$

$F_{\max} = 6.25kN$ (3s.f.)

3)

Firstly, we note that the weighing scale only measures the force in the vertical direction.

Entering the frame of the person, the forces acting on him is the fictious force (caused by change in frame), and weight.

Thus, the effective force he exerts on the weighing scale is balanced by the normal force.

$mg - ma\sin\theta = N$

$m(g - a\sin\theta) = N =$ reading on scale

4a)

By conservation of energy, and considering that the projectile has initial mass m.

$\frac{1}{2}m\left(v\sin\theta\right)^2 = mgh$

After simple algebraic manipulation,

$h = \frac{v^2\sin^2\theta}{2g}$ (Shown)

4b)

Taking the y-component of the two objects taking time t_2 to drop as v_2,
and taking the y-component of the object taking time t_1 to drop as v_1,
By conservation of linear momentum,

$mv_1 + 2mv_2 = 0$

$v_1 = -2v_2$

Taking the point of explosion as zero, and the upwards direction as positive,

$-h = -v_1 t_1 - \frac{1}{2} g t_1^2$

$-h = v_2 t_2 - \frac{1}{2} g t_2^2$

Substituting $v_1 = -2v_2$,

$-h = 2v_2 t_1 - \frac{1}{2} g t_1^2$

$\Rightarrow -ht_2 = 2v_2 t_1 t_2 - \frac{1}{2} g t_1^2 t_2$ and $-h(2t_1) = v_2 t_2 (2t_1) - \frac{1}{2} g t_2^2 (2t_1)$

Summing the equations up,

$h(2t_1 - t_2) = \frac{1}{2} g t_2^2 (2t_1) - \frac{1}{2} g t_1^2 t_2 = \frac{1}{2} g t_1 t_2 (2t_2 - t_1)$

$h = \frac{g t_1 t_2 (2t_2 - t_1)}{2(2t_1 - t_2)}$

5a)

On Earth, the atmosphere contains air molecules which produces a drag resistive force against any moving object. The force is directly proportional to the velocity of the moving object. As such, the acceleration of the object becomes

$$a = \frac{dv}{dt} = g - \frac{b}{m} v$$

where m is the mass of the object, *b* the coefficient of drag (which is proportional to area and shape of the object), and *g* the gravitational constant.

Thus, it is clear that the mass of the object plays an important role in the acceleration of the object as it falls. The larger the mass, the smaller the acceleration. Thus the metal ball and feather which have different masses, will have different accelerations.

5b)

There is electrical attraction between the molecules when they collide with each other, and hence, the collision becomes inelastic. This inelasticity reduces their post-collision velocities and hence deviation from the ideal gas law. The wall of the container is much larger than the molecules, hence when the molecules hit the wall, the electrical attraction is minimal and the collision can be assumed as perfectly elastic, and thus the ideal gas law holds.

5c)

The atmosphere does not have a uniform index of refraction. The light coming from the two ends of the sun will enter the atmosphere at different positions, and as such, they will be refracted differently. This causes the sun to appear contracted vertically.

5d)

The armature can be treated as an electrical source, and the field windings can be considered as a resistor. If they are in series, and they are connected to a load in series, the overall equivalent resistance will be higher, and thus it slows down faster as more energy is dissipated per unit time. If they are in parallel, the equivalent resistance of the entire system will be lower, and thus the energy dissipated per unit time is less.

6)

7a)

The radio waves can take multiple paths to reach the ship.
As such, this problem can be treated as a double-slit interference problem (Llyod's mirror) with the water surface acting like a mirror.
There will be a phase change of π after one of the paths is reflected from the water surface (water is optically denser than air)
Hence, we can write the interference equation as

$$m\lambda = \frac{dL}{y} \text{ (destructive interference)}$$

with m being the order of interference, d the distance between the two sources (image source in this case), L the vertical distance the ship's receiever is at from the water surface, and y the distance of the ship to the source.
The first minimum the ship receives occurs when m = 1,
Substituting the results,
y = 533m

7b)

As we do not know from the question whether the ship is moving in the perpendicular direction of the two sources, or the parallel direction, we will hence solve for the two cases.

Parallel direction:
m = 1, m = 2 for destructive interference (from 7a)

$$\text{distance travelled } = \Delta L = \frac{yd}{2\lambda}$$

$$speed = \frac{\Delta L}{50s} = \frac{\text{yd}}{2\lambda(50s)} = 142ms^{-1}$$

Perpendicular direction:
m = 1, m = 2 for destructive interference (from 7a)

$$\text{distance travelled } = \Delta y = \frac{L\lambda}{d}$$

$$speed = \frac{\Delta y}{50s} = \frac{L\lambda}{d(50s)} = 0.015ms^{-1}$$

8)

$$[\mathrm{P}] = \frac{\mathrm{kgms^{-2}}}{\mathrm{m^2}} = kg \cdot m^{-1} \cdot s^{-2}$$

$$[V] = m^3 = [b]$$

$$[\frac{a}{V^2}] = \frac{[a]}{[V]^2} = [P]$$

$$[a] = [P][V]^2 = kg \cdot m^5 \cdot s^{-2}$$

For the equation, the critical point occurs with the conditions such that,

$$\left(\frac{\partial \mathrm{P}}{\partial V}\right)_T = 0 \text{ and } \left(\frac{\partial^2 P}{\partial V^2}\right)_T = 0$$

ie. keeping the temperature as a constant while differentiating V with respect to P.

This is because the critical point is a point of horizontal gradient and a inflection point on the P- V graph.

Thus,

$$\left(\frac{\partial \mathrm{P}}{\partial V}\right)_T = \frac{-RT}{(V-b)^2} + \frac{2a}{V^3} = 0$$

$$\left(\frac{\partial^2 P}{\partial V^2}\right) = \frac{2RT}{(V-b)^3} - \frac{6a}{V^4} = 0$$

Rewriting these equations by replacing V with V_c and T with T_c,

$$\frac{-RT_c}{(V_c-b)^2} + \frac{2a}{{V_c}^3} = 0$$

$$\frac{2RT_c}{(V_c-b)^3} - \frac{6a}{{V_c}^4} = 0$$

We will now need to solve the above two simultaneous equations,

$$\left(\frac{-RT_c}{(V_c-b)^2} + \frac{2a}{{V_c}^3}\right)\frac{2}{V_c - b} = \frac{-2RT_c}{(V_c-b)^3} + \frac{4a}{{V_c}^3\left(V_c - b\right)} = 0$$

$$\Rightarrow -\frac{6a}{{V_c}^4} + \frac{4a}{{V_c}^3\left(V_c - b\right)} = 0 \rightarrow \frac{6a}{V_c} = \frac{4a}{\left(V_c - b\right)}$$

Simplifying, $V_c = 3b$.

$\Rightarrow$ Substituting back, $\dfrac{-RT_c}{(3b-b)^2} + \dfrac{2a}{\left(3b\right)^3} = 0$

$T_c = \dfrac{8a}{27bR}$ (Shown)

Substituting T_c and V_c into the original Van der Waals equation,

$$P_c = \frac{RT_c}{V_c - b} - \frac{a}{{V_c}^2} = \frac{R\left(\frac{8a}{27bR}\right)}{3b-b} - \frac{a}{\left(3b\right)^2}$$

$P_c = \dfrac{a}{27b^2}$ (Shown)

9a)

By Newton's Second Law,

In this question, the variable one should be concerned with should be Δs
and not Δx, which is essentially a constant. It was left there to confuse the
solver. Now, looking at the problem, the force that is causing this mass to be
pushed back is caused by $A \cdot \Delta s$. We will replace Δs with s from now. Consider the mass of the air is m,
By Newton's second law,

$$\sum F = ma$$

The air that is further away from the equilibrium is at a distance $x + s$ away. With x as a constant,
the acceleration of the air will be $\ddot{s}$, hence:

$$m\ddot{s} = \Delta P \cdot A = -BA\frac{\Delta V}{V}$$

$$\frac{\Delta V}{V} = \frac{As}{A\Delta x} = \frac{s}{\Delta x}$$

$$\therefore m\ddot{s} + BA\frac{s}{\Delta x} = 0$$

which is a simple harmonic motion equation!

$$\ddot{s} + \frac{BA}{m\Delta x}s = 0$$

The above SHM equation has an angular frequency ω such that,

$$\omega^2 = \frac{BA}{m\Delta x}$$

$$\Rightarrow (\Delta x)^2\omega^2 = v^2 = \frac{BA\Delta x}{m} = \frac{BV}{m} = \frac{B}{\rho}$$

$$\therefore \rho v^2 = B$$

9b)

For a one-sided open ended tube, the equation of the wavelength to its length is

$$L = \frac{1}{4}\lambda \text{ (fundamental mode of vibration)}$$

$$f\lambda = \frac{\lambda}{T} = v$$

$$T = \frac{\lambda}{v} = \frac{4L}{v}$$

Modifying these terms to the question,

L = average radius of the star = 9.0×10^{-3} solar radius

From part a), we know that $\rho v^2 = B$

$$\therefore v = \sqrt{\frac{B}{\rho}} = \sqrt{\frac{1.33 \times 10^{22}\,\text{Pa}}{10^{10}\,\text{kgm}^{-3}}} = 1.15 \times 10^6 ms^{-1}$$

$$T = \frac{4L}{v} = \frac{4(9.0 \times 10^{-3} \times 6.955 \times 10^8)m}{1.15 \times 10^6 ms^{-1}} = 21.8s \text{ (3s.f.)}$$

10.

Let the wavelength of the original photon be λ

By Compton's scattering,

$\Delta\lambda = \frac{h}{M_0 c}(1-\cos\theta)$, where θ is the angle of deflection from the incident plane

Hence, by conservation of energy,

$\frac{hc}{\lambda} + M_0 c^2 = \frac{hc}{\lambda + \Delta\lambda} + M_0 c^2 + K_e$, K_e is the kinetic energy of the recoiling electron

$$K_e = \frac{hc}{\lambda} - \frac{hc}{\lambda + \frac{h}{M_0 c}(1-\cos\theta)}$$

To maximize K_e, we have to minimize $\frac{hc}{\lambda + \frac{h}{M_0 c}(1-\cos\theta)}$

$\Rightarrow$ We will need to maximize $\lambda + \frac{h}{M_0 c}(1-\cos\theta)$, which occurs when $\theta = \pi, \cos\theta = -1$

$$K_{e,\max} = E - \frac{\frac{hc}{\lambda}}{1 + \frac{2hc}{M_0 \lambda c^2}} = E - \frac{EM_0 c^2}{M_0 c^2 + 2E} = \frac{EM_0 c^2 + 2E^2 - EM_0 c^2}{M_0 c^2 + 2E}$$

$= \frac{2E^2}{M_0 c^2 + 2E}$ (Shown)