SJPO 2024 Answer Key

Foreword by solutions author: This set of solutions is for the postponed SJPO in July 2024. This paper has a higher proportion of algebraic questions than previous years, though they are not uncharacteristically difficult. Some careful thought when applying knowledge from the SJPO syllabus will suffice in almost all the questions — many are in fact only 1 to 2 lines of working.

1. **D** By drawing a free-body diagram, it can be observed that the vertical component of the normal force must balance gravity. Hence, $N \sin\theta = mg \implies N = mg/\sin\theta$

2. **B** We can deduce that the speed of the northbound train is v as the tree is stationary in the world frame. Since the relative velocity between the trains is u, the southbound train's speed is simply $u - v$.

3. **E** Intuitively, one can consider the point where it has its average kinetic energy, as a constant acceleration would be analogous to considering a constant rate of work done by distance. Alternatively, we can solve for the midpoint velocity v_m in:

$$2as = v^2 - v_m^2 = v_m^2 - u^2$$

   which yields the same answer $v_m = \sqrt{\frac{1}{2}(u^2 + v^2)}$.

4. **C** We can intuitively consider the force evenly distributed over all blocks to give F/n. Alternatively, the force can be found by finding the (equal) acceleration and observing that the normal force between the two blocks is the only force on it.

5. **A** This is a less conventional application of the standard "system method" of solving pulley questions. By considering the chain as a connected system, the only force causing movement is the gravitational force $W = mg/4$ on the hanging edge. Since the total mass of the chain is m, the initial acceleration is $a_{\text{sys}} = g/4$.

6. **B** We must balance the weight from the previous part with the friction force, which is $f = \mu(3m/4)g$ since only 3/4 of the chain is resting on the table. Hence, $\mu = 1/3$.

7. **C** This is a confusing question as the statements may apply to a certain frame of reference (which may be accelerating) and requires some precision:
   - (A) Alice's FBD cannot be correct as the force to the right is not the centripetal force (which is a net force). If a force is drawn in such a manner, it would refer to the *centrifugal* force, which is fictitious. This would be applicable to make it in equilibrium in the frame of reference of the ball.
   - (B) See explanation in (A) — centripetal force is not fictitious.
   - (C) This is correct in the world frame of reference. Note that it is not in equilibrium due to the inward acceleration from circular motion.
   - (D) They do not form an action-reaction pair — the forces are of different type, different magnitude and not in opposing directions or on opposite objects.
   - (E) See (A) for the same reason why Alice's is invalid.

8. **C** This can be solved by moving into the frame of reference of the bus, which must have acceleration at $\theta = 35°$ to the right of the vertical. Hence, the original acceleration of the bus must be to the left and of magnitude $g\tan\theta = 6.9\ \text{m s}^{-2}$.

9. **D** We balance torques using the upper step's corner as the point of reference (since it pivots about this point). Since F is minimised when it is directed to the top of the box, we have $F(2L/3) = mg(L/2)$, which we can rearrange to give $F = 3/4Mg$.

10. **B** We shall break the motion into parallel and perpendicular components to reason out the motion more easily.
    - I. By considering the perpendicular components, this is analogous to a vertical bouncing that repeats identically. Hence, the time between bounces is in fact identical. (False)
    - II. As the ball accelerates in the parallel direction, d_{II} increases. (True)
    - III. Similarly to statement I, d_{III} is constant. (True)
    - IV. Similarly to statement II, the angle will decrease as the parallel component of the velocity is always in-

creasing. (False)

11. **B** We note that in the accelerating frame of reference of freefall, the arrow travels at constant velocity towards the target for a distance s. Hence, the time taken is $t = s/u$. We can then find $d = \frac{1}{2}gt^2 = \frac{gs^2}{2u^2}$.

12. **C** Let the speed of the ball be v. By conservation of momentum we have $mv = M_1 v_1$ and $mv = (m + M_2)v_2$, hence $v_1/v_2 = M_1/(M_2 + m)$.

13. **A** Each rod has centre of mass at length $l/2$ from the tip, hence the change in GPE of each rod is $mgl/2(\sin\alpha - \sin\beta)$. Substituting and multiplying by 6 gives 0.95 J.

14. **E** Due to the constrained motion, the centre mass must have 0 velocity when the top two are about to collide — the symmetry only allows for downward motion, and the angular motion will have no vertical component just before collision. Hence, a direct conservation of energy gives $mg\ell = \frac{1}{2}(2m)v^2$, or $v = \sqrt{g\ell}$. The relative velocity is then $2v = \sqrt{4g\ell}$.

15. **E** The work done by resistive forces will simply be the change in total potential energy. We have $mg = kx_0$ and $(m + M)g = kx_f$. Hence, the change in potential energy is

$$\Delta E = \frac{1}{2}kx_f^2 - \frac{1}{2}kx_0^2 - (m + M)g(x_f - x_0) = -\frac{(M - m)^2 g^2}{2k}$$

16. **E** We need torques to be balanced about the tip of the left block — hence, $D = 4(L - D) \implies D = 4/5L$.

17. **D** The ant will be launched upwards (in the diagram) when it jumps, so it will only need to clear the distance to the edge of the turntable there (which is $4/5R$). Hence,

$$t = \frac{\frac{4}{5}R}{\left(\frac{3}{5}R\right)\omega} = \frac{4}{3\omega}$$

18. **D** Consider the square formed between the endpoints and the inner corner. The diagonal is of radius R and its side length is $R - L = R/\sqrt{2}$. Hence, $R = (2 + \sqrt{2})L$, and by circular motion,

$$\frac{v^2}{R} = \mu g \implies v = \sqrt{(2 + \sqrt{2})\mu g L}$$

19. **A** We must have the maximum acceleration never exceed g such that there will never be a point where the string goes slack. Hence, $\omega^2 A = g$ where $\omega = \frac{2\pi}{T}$ can be substituted to give $A = 1.6$ cm.

20. **D** Let the impulse be J. The linear momentum hence increases by J and angular momentum by JR about the centre of mass. Hence,

$$\frac{1}{2}MR^2\omega = mvR$$

which implies $R\omega = 2v$. Hence, $v_{\text{left}} = 3v$ and $v_{\text{right}} = -v$, which gives a ratio of 3.

21. **D** The ball rolls without slipping, i.e. $v = r\omega$. This means that the rotational KE is 2/5 of the translational KE, i.e. the rotational KE is 5/7 of the total KE. Hence, $d = 2/7h$ as this deficit remains as rotational KE.

22. **B** We consider each:

    I. True — this is an application of Kepler's 3rd Law which gives $\sqrt{\frac{4\pi^2 a^3}{GM}}$ where a is the semimajor axis.

    II. True — the total energy is $-\frac{GMm}{2a}$.

    III. False — assuming constant total energy (due to the constant major axis), the angular momentum controls the eccentricity of the circle/ellipse. As angular momentum decreases, the ellipse becomes more "squished" (more eccentric). Hence, Satellite Y has less angular momentum than Satellite X.

    IV. False — by COE, we should have $v_1 > v_2 > v_3$ instead. Option III is tricky to understand and is somewhat of a fringe topic of SJPO. By getting the rest of I, II and IV correct, a process of elimination can also yield the correct answer of B.

23. **B** We simply conserve energy as their separation decreases from $8R$ to $2R$:

$$-\frac{GM^2}{8R} = -\frac{GM^2}{2R} + 2 \cdot \frac{1}{2}mv^2 \implies v = \sqrt{\frac{3}{8}gR}$$

where $g = \frac{GM}{R^2}$.

24. **A** We can apply Bernoulli's equation to the pipe, with:

$$\frac{1}{2}\rho v^2 = \rho g h$$

which gives $v = 3.1$ m s$^-1$.

25. **D** By Archimedes' principle, we see that for any vertical displacement x from the equilibrium, we have a net force $\rho_2 g x \ell^2$, which gives acceleration $\frac{\rho_2 g}{\rho_1 \ell}x$. Hence, $\omega = \sqrt{\frac{\rho_2 g}{\rho_1 \ell}}$ and $T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{\rho_1 \ell}{\rho_2 g}}$.

26. **C** We can see that the critical angle must minimally be 60°, hence $n \geq \frac{1}{\sin 60°} = \frac{2}{\sqrt{3}}$.

27. **C** The mirror is of length $L = 1.6 - 0.9 - 0.4 = 0.3$ m. By mirroring the person to the right of the mirror at an equal distance and drawing lines of projection from the eyes of the person, we see that by similar triangles, $0.3 \times 2 = 0.6$ m of the image can be seen — this corresponds to $0.6/1.6 \approx 38\%$ of his body.

28. **A** We can approximate the rays from the sun as parallel, and hence will converge towards the focal point. By similar triangles, the diameter of the bright spot can be found by scaling down the diameter of the lens to give $d' = 10 \times \frac{20-15}{20} = 2.5$ cm.

29. **E** We consider the system as a fully symmetric cube with all $+q$ corners, superimposed with a charge of $-2q$ at the bottom left. Since the symmetric system causes no net field at the centre, we only need to consider the $-2q$ charge, which provides a field of:

$$\frac{1}{4\pi\varepsilon_0}\frac{2q}{\left(\frac{\sqrt{3}}{2}a\right)^2} = \frac{1}{4\pi\varepsilon_0}\frac{8q}{3a^2}$$

30. **E** By Gauss' law, the flux passing through all 6 faces must be equal to Q/ε_0, hence that through one face must be $Q/6\varepsilon_0$. We multiply by the charge density $\sigma = q/a^2$ to get $F = Qq/6\varepsilon_0 a^2$.

31. **A** By Gauss' law, the flux will only depend on the enclosed charge $+Q$ — we can hence ignore the outer shell entirely. Hence, a direct application of Coulomb's law will give $\frac{Qq}{4\pi\varepsilon_0}\left(\frac{1}{a} - \frac{1}{b}\right)$.

32. **B** The 3Ω and 6Ω parallel resistors have an effective resistance of 2Ω — by potential divider rule, the voltage across the pair is $20 \times \frac{2}{2+4+4} = 4$ V. Hence, $P = \frac{V^2}{R} \approx 5.3$ W.

33. **C** Notice that by symmetry, the two unmarked vertices will have equal potential. This means we may eliminate the resistor between them, leaving two branches of $2R$ and one branch of R all in parallel. This gives an effective resistance of $\left(\frac{1}{2R} + \frac{1}{2R} + \frac{1}{R}\right)^{-1} = \frac{R}{2}$

34. **C** We first charge the 10 and 20μF capacitors, which are in series with an effective capacitance $C = \left(\frac{1}{10} + \frac{1}{20}\right)^{-1} = \frac{20}{3}\mu$F. This leaves a charge of $Q = CV = \frac{2000}{3}\mu$C on each capacitor. After discharging via B, the voltage must equalise — we solve:

$$Q_{20} + Q_{40} = \frac{2000}{3}$$
$$\frac{Q_{20}}{20} = \frac{Q_{40}}{40}$$

to get $Q_{40} \approx 440\mu$C.

35. **B** A simple argument can be made by considering the point X, as well as another point on A above B. The point above B will a $\vec{B}$ field out of the page and a Lorentz force to right, while X will have a $\vec{B}$ field into the page and a Lorentz force to the left, causing it to rotate clockwise. Alternatively, an intuitve idea is that the currents wish to "align" as like currents attract.

36. **B** By a direct application of any cross product method (Fleming LH rule / RH grip rule for cross products), the Lorentz force will always be perpendicular to the velocity and in the plane; initially, this will be roughly southeast and will cause it to move in a clockwise circle.

37. **C** Imagine a path comprising two lines on either side of an infinite charged sheet — using this to apply Ampere's law, we expect that each of them will have a field of $B = \frac{1}{2}\mu_0 I$ per unit length (as the enclosed current gives $\mu_0 I$ for the two lines). This gives a total force of $\frac{1}{2}\mu_0 I^2$ between the two sheets.

    (Additional remarks: the thought process is similar to an infinitely charged sheet for Gauss' law)

38. **D** While this is traditionally a calculus-based question, there is a somewhat simple argument — consider one revolution of the disc, which sweeps out an area of πR^2 in one period T. Since we have $T = 2\pi/\omega$ and the rate of change of area can be taken to be constant, we have $\varepsilon = \pi R^2 B\omega/2\pi = \frac{1}{2}R^2\omega B$.

39. **E** The speed that waves travel in a string is given by $v = \sqrt{T/\mu}$ where T is the tension in the string and μ is the linear mass density. Since the latter is the same for the two string segments and the tension is in the ratio of $T_1/T_2 = (m_1 + m_2)/m_2$, our wavelength ratio is $\sqrt{(m_1 + m_2)/m_2}$.

40. **B** We consider each statement:

    I. True — consider that for a wave, we must have $v = \omega/k$ be the same for all modes, hence $\omega' = 2\omega$.

    II. False — This would mean the lowest wavelength would be $4L$, as we have one node and one antinode. However, our smallest wavelength suggested by the first term is $2L$ (by using $k = 2\pi/\lambda$).

    III. True — this is a direct result of statement I too.

    IV. False — the energy of a wave is proportional to the square of amplitude.

41. **D** In the parallel case we have perfect constructive interference, which increases the amplitude to $A = 2A_0$. In the perpendicular case, the vector sum gives $\sqrt{2}A_0$. Hence, the resulting amplitude is $A/\sqrt{2}$.

42. **E** The bright and dark fringes are a result of constructive and destructive interference respectively — this means that we wish to directly add/subtract their amplitudes, which are each proportional to $\sqrt{I}$. Hence, our resulting amplitude ratio $\dfrac{A_1'}{A_2'} = \dfrac{\sqrt{I_1} + \sqrt{I_2}}{\sqrt{I_1} - \sqrt{I_2}}$ and squaring it gives our answer.

43. **B** We have $d\sin\theta = m\lambda$, where $m = 2$ will be the last peak (-2,-1,0,1,2 are our 5 peaks). Hence, $\lambda = d/2$ is the maximum. Note that this peak is at $\theta = \pi/2$ which is technically not observable, but a minute change in any values will bring it onto the screen.

44. **C** Letting the final temperature be T_f in Celsius, we have $c_w m_w(30 - T_f) = c_w m_i T_f + m_i l_f$. Solving with substitutions from the data table, we get ≈ 8.1°C.

45. **A** This question is most easily solved by the circuit analogy with effective resistance L/kA. Letting the length and cross-sectional area of the system be $2L$ and $2A$ respectively, our blocks have resistance $R_1 = L/400A$, $R_2 = L/400A$ and $R_3 = L/600A$ each. Since R_2 and R_3 are in parallel, they have an effective resistance of $R_{2//3} = L/1000A$, which implies the midpoint is at 2/7 of the potential difference between T_{hot} and T_{cold} (by potential divider rule). Hence, $T_{\text{centre}} = 20 + (100 - 20) \times 2/7 = 42.9$°C (closest answer is 40°C).

46. **A** After expansion, the two rods will have lengths $L_1 = L_0(1 + \alpha_1)\Delta T$ and $L_2 = L_0(1 + \alpha_2)\Delta T$ Since their centres are spaced a distance d apart from each other, we can say that $L_2 = R\theta$ and $L_1 = (R + d)\theta$. We may subtract to get $\theta = \frac{L_0}{d}(\alpha_1 - \alpha_2)\Delta T$.

    This problem can be guessed by considering special cases like $\alpha_1 = \alpha_2$.

47. **C** The input and output power of the orbiting planet must be the same. For the input, we consider the proportion of the power output from the sun, which is roughly $\pi r^2/4\pi d^2$. Hence, $\dfrac{r^2}{4d^2}(4\pi R^2\varepsilon\sigma T_s^4) = 4\pi r^2\varepsilon\sigma T_p^4$. Rearranging, we get $\dfrac{T_p}{T_s} = \sqrt{\dfrac{R}{2d}}$.

48. **A** At equal temperatures and pressures at the start, we have $N/2$ molecules in each. Following the temperature change, we have $N_1 = 7N/12$ and $N_2 = 5N/12$ to be able to maintain the same pressure and volume (since $PV = nRT$). Hence, a net transfer of $N/12$ occurs.

49. **C** Notice the horizontal lines are isothermal, where work done by the gas is given by $nRT \ln \frac{V_f}{V_i}$ (as it should be positive when V increases), and the vertical segments are isovolumetric, where no work is done. Hence, the net work done is $nRT_0 \ln 3$ by the gas.

50. **E** As the balloon is perfectly insulating, we have $Q = 0$ as no heat is transferred — this is hence an adiabatic process. We have a new pressure of $P = \rho g h$, where $PV^\gamma = P_0 V_0^\gamma$. This can be rearranged to use the temperature, which is $P^{1-\gamma} T^\gamma = P_0^{1-\gamma} T_0^\gamma$. Substituting, we get $T \approx 364$ K.

For clarifications on solutions, email Paul (problem & solution author) via contact@pd-stem.com. This has been written independently of IPS.