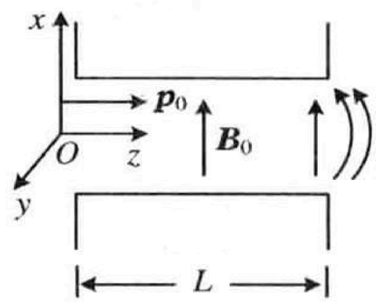
APhO 2005 T2B: Magnetic lens (improved)

This is adapted from a Chinese textbook (物理学难题集萃，舒幼生 et al.). The original APhO problem was worded really poorly and used some weird artificial assumptions that aren't really necessary or correct. Don't reference the original problem, as it will spoil some parts of the solution (even the title of the problem is a spoiler!). The difficulty of this problem is around a EuPhO T2.

Define a coordinate system as shown. We have two identical cuboidal permanent magnets with uniform magnetisation along the x-axis. The length of the magnets in the z-direction is L. The lengths of the magnets in the $\pm x$ direction and $\pm y$ direction are much larger than L. The magnets are placed with opposing poles close to each other (separation $\ll L$) to create a magnetic field with magnitude B_0 at the center of the cross-section.



A stream of charged particles, each with charge q, is ejected at $y = z = 0$ at various positions along the x-axis. The particles each have an initial momentum of $\boldsymbol{p_0} = p_0\hat{\boldsymbol{z}}$, where $p_0 \gg qLB_0$. Show that the charged particles are focussed to a point, and find the z-coordinate of this point.

Ignore mutual interactions between the charged particles. The particles are moving at a nonrelativistic speed.

Solution

The reason there is such a focussing phenomenon is because of the fringe fields of the magnet, which provides a component of the magnetic field in the z-direction. This can interact with the y-component of the velocity of the particles.

The cyclotron frequency is $\omega = qB_0/m$, so the angular deflection of the particle in the y-z plane along the y direction is

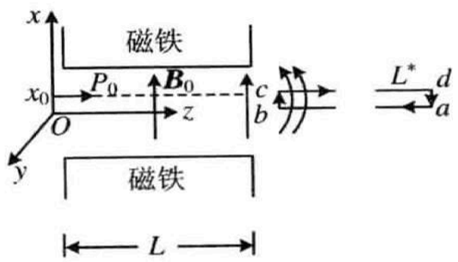
$$\theta_y = \frac{qB_0}{m}t \approx \frac{qB_0}{m}\frac{L}{v_0} = \frac{qB_0 L}{p_0} \tag{1}$$

Note that the total magnetic flux of the fringe field is much less than the flux in the space between the magnets, so we can ignore the fringe field when finding θ_y. However, the focussing effect is only due to the fringing field: see the remark below. Using Newton's 2nd law and the small angle approximation,

$$m\frac{\mathrm{d}v_x}{\mathrm{d}t} = qv_y B_z \approx qv\theta_y B_z \tag{2}$$

$$\therefore v_x = \frac{qv\theta_y}{m}\int B_z\,\mathrm{d}t = \frac{q\theta_y}{m}\int B_z\,\mathrm{d}z \tag{3}$$

This integral can be found using Ampere's law. Create an Amperian loop as shown. The extent of the fringing field is on the order of L, but we will soon see that $f \gg L$, so we can put the points a and d essentially at infinity. For a particle with an initial x-coordinate of x_0,



$$\oint \boldsymbol{B}\cdot\mathrm{d}s = B_0 x_0 + \int B_z\,\mathrm{d}z = 0 \tag{4}$$

$$\therefore \int B_z\,\mathrm{d}z = -B_0 x_0 \quad \Rightarrow \quad v_x = -\frac{q\theta_y}{m}B_0 x_0 \tag{5}$$

Using similar triangles,

$$\frac{x_0}{f} \approx \frac{x_0}{f-L} = -\frac{v_x}{v_0} \tag{6}$$

$$\boxed{\therefore f = \frac{p_0^2}{q^2 B_0^2 L} \tag{7}}$$

Remark: the integral of B_z in the space within the magnets is negligible compared to that of the fringe field, which can be seen by creating an Amperian loop with one edge at the center of the cross-section and another at the edge. This means that almost all of the focussing occurs in the fringe field, while most of the y-deflection (θ_y) occurs in the space between the magnets.