General Instructions for Theoretical Examination

Before the Exam

1. The allocated time for theoretical examination is 5 hours for answering 3 questions that carry 10 points each.
2. Please ensure that your student code matches the one on your desk. Place your ID / tag on the top right corner of the desk.
3. Kindly ensure that you have the following items on the desk:
   (i) A sealed envelope containing the Question Papers and Answer Scripts.
   (ii) A labelled envelope.
   (iii) Pens/relevant stationery (You need to bring in your own calculator).
   If you find that the items are incomplete, kindly raise your hands.
4. You must not open the sealed envelopes before the start of the examination.

During the Exam

1. You must use the dedicated answer scripts provided for writing your answers. Write your answers only in boxed answer scripts. If you need more boxed answer scripts, please raise your hand and ask an invigilator. Clearly write your student code and write which question you are attempting on each of your extra requested answer scripts.
2. **There are additional blank draft papers. These will not be marked.**
3. Try to be as concise as possible in your answers: use equations, logical operators, and sketches to illustrate your thoughts whenever possible. Avoid the use of long sentences.
4. Use an appropriate number of significant figures when stating numbers.
5. You are not allowed to leave your working place without permission. If you need any assistance (malfunctioning calculator, need to visit a restroom, insufficient Draft Papers, etc), please draw the attention of the invigilator by raising your hand.

At the end of the Exam

1. At the end of the examination, you must stop writing immediately.
2. Put all the Answer Scripts into the labelled envelope and seal it, including any blank Answer Scripts. It is your responsibility to ensure all Answer Scripts are inside the envelope and submitted to the invigilator.
3. Put question papers and draft papers into another envelope. You are not allowed to take any sheets of paper out of the examination hall.
4. Wait at your table in silence till your envelopes are collected. Once all envelopes are collected, your guide will escort you out of the examination area.

You must use the dedicated Answer Scripts provided for writing your answers. Write your answers only in boxed answer scripts. If you need more boxed answer scripts, please raise your hand and ask an invigilator. Clearly **write your student code** and **write which question number you are attempting** on each of your extra requested answer scripts.

There are additional blank draft papers. These will not be marked.

Physical Constants

Acceleration due to gravity	$g = 9.81\ \mathrm{m\,s^{-2}}$
Boltzmann constant	$k_B = 1.38 \times 10^{-23}\ \mathrm{J\,K^{-1}}$
Current Mass of the Sun	$M_S = 2.00 \times 10^{30}\ \mathrm{kg}$
Current Radius of the Sun	$R_S = 7.00 \times 10^{8}\ m$
Average Mass of a Galaxy	$M_{galaxy} = 1.5 \times 10^{12}\ M_S$
Magnitude of the electron charge	$e = 1.60 \times 10^{-19}\ \mathrm{C}$
Mass of the electron	$m_e = 9.11 \times 10^{-31}\ \mathrm{kg}$
Mass of the proton	$m_p = 1.67 \times 10^{-27}\ \mathrm{kg}$
Mass of the neutron	$m_n = 1.67 \times 10^{-27}\ \mathrm{kg}$
Permeability of free space	$\mu_0 = 4\pi \times 10^{-7}\ \mathrm{T\,m\,A^{-1}}$
Permittivity of free space	$\varepsilon_0 = 8.85 \times 10^{-12}\ \mathrm{F\,m^{-1}}$
Intrinsic impedance of free space	$Z_0 = 120\pi\ \Omega$
Planck's constant	$h = 6.63 \times 10^{-34}\ \mathrm{J\,s}$
Speed of light in vacuum	$c = 3.00 \times 10^{8}\ \mathrm{m\,s^{-1}}$
Stefan-Boltzmann constant	$\sigma = 5.67 \times 10^{-8}\ \mathrm{W\,m^{-2}\,K^{-4}}$
Universal Gravitational constant	$G = 6.67 \times 10^{-11}\ \mathrm{N\,m^{2}\,kg^{-2}}$

Geometry of Water Fountain

Introduction

Water sprayers are found at various places such as agriculture farms, green parks or urban areas for functional purposes or for aesthetic art installation. Consider a hemispherical shaped fountain sprayer of radius r at the height h from the ground as shown in a cross-sectional diagram in **Fig. 1**. Let the radius of the hemisphere to be small compared to the range R, hence can be treated as a point source. However there are $\rho(\theta)$ number of holes per unit area at angle θ. The water spurts in all directions at the same initial velocity v_o.

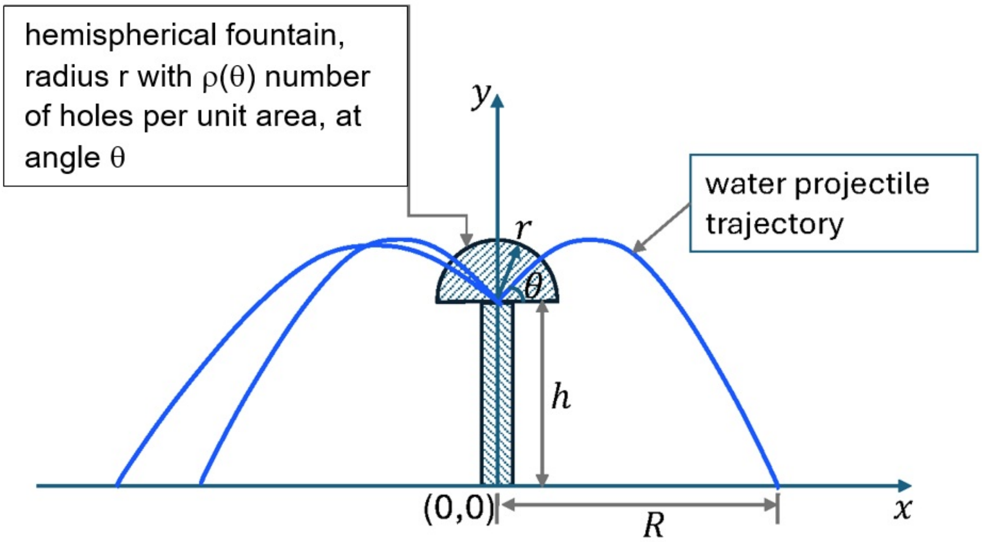

Fig. 1: Schematic cross section of a hemispherical shaped fountain sprayer.

Part A: Uniformly Distributed Holes on the Surface of the Hemisphere (6.0 points)

First consider the holes distribution on the surface of the hemisphere to be uniform (i.e. $\rho(\theta)$ is a constant).

A.1	Express the instantaneous position $x(t)$ and $y(t)$ of a water element (can be treated as a stream of particles) launched at velocity v_o at angle θ.	**0.5pt**

A.2	Derive the water trajectory relation $y = y(x, \theta)$ and express in a same type of trigonometric function (e.g. in terms of sine, cosine or tangent ONLY). Note that $\theta > 0$.	**1.0pt**

A.3	The envelope is the minimal boundary which encloses all water trajectories (i.e. no water will be found beyond this boundary). Derive the equation for the envelope for this system. Express your answer as y(x). Sketch the envelope over the trajectories.	**3.0pt**

A.4	Calculate the range $R = R(\theta)$ as a function of angle θ.	**1.0pt**

A.5	Letting $h \to 0$ in the range obtained in **A.4**, show that $R = \frac{v_o^2}{g} \sin\ 2\theta$.	**0.5pt**

Part B: Non-Uniformly Distributed Holes on the Surface of the Hemisphere (4.0 points)

Now, consider the area of holes per unit area $\rho(\theta)$ to be non-uniformly distributed and depended on the angle θ.

B.1	Using the results from part **A.4** above, calculate the elemental area dA_W of an annulus (ring) of radius R and width dR of the water hitting the ground. Express your answer in R, R' and dθ. [hint: consider R'=$\frac{dR}{d\theta}$]	**1.0pt**

B.2	Determine the relation for $\rho(\theta)$ that gives a uniform spray coverage pattern on the ground by considering elemental area on the hemisphere in proportional to the elemental area of the water spray on the ground. Express your answer in terms of R and its derivative. [hint: consider condition where $R\prime(\theta) > 0$].	**1.0pt**

B.3	Letting $h = 0$, calculate explicitly the relation for $\rho(\theta)$ that gives a uniform spray coverage pattern for $\theta < 45^\circ$.	**2.0pt**

Snell's Law

Introduction

When light is incident upon a dielectric interface, it will be reflected and refracted, depending on the incident angle of the light and the refractive indices of the dielectric media as shown in **Fig. 1**. The refraction of light is governed by the Snell's law,

$$n_1 \sin \phi_1 = n_2 \sin \phi_2 \tag{1}$$

where n_1 and n_2 are the refractive indices of the lower and upper parts of the boundary, ϕ_1 and ϕ_2 are the angles that the light ray makes with the normal of the boundary.

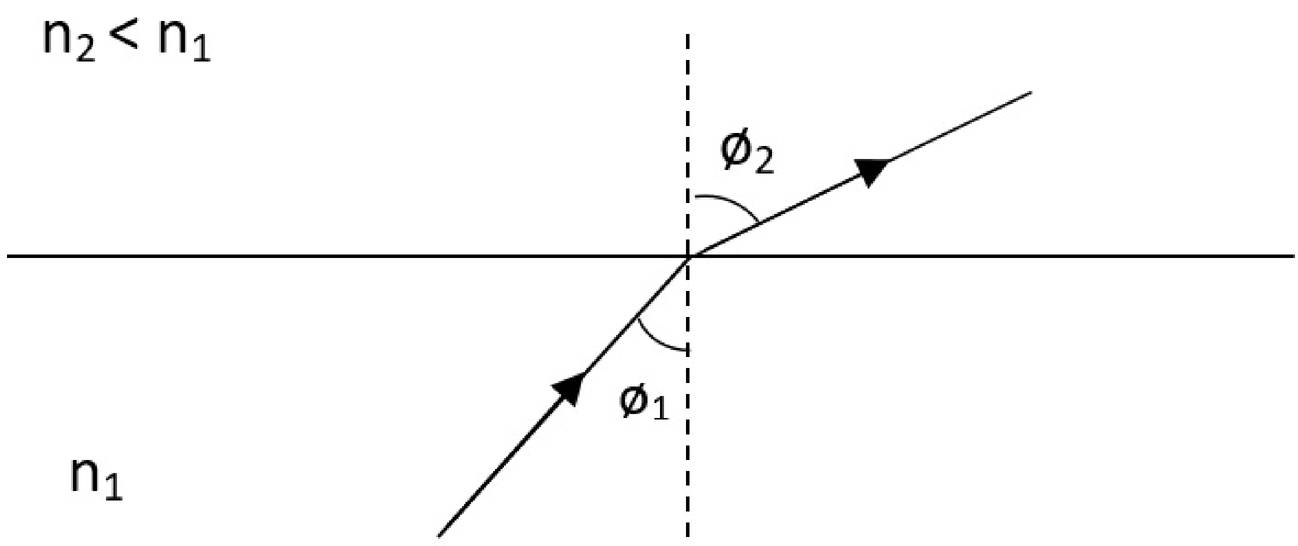

Fig. 1: Refraction of light from a dielectric medium with refractive index, n_1, to a second dielectric medium with refractive index, n_2.

Part A: Light Propagation Through a Semi-Sphere (1.5 points)

Fig. 2 illustrates how light that consists of two rays with different colour, a and b, is incident along the radius of a semi-sphere with refractive index n_x, in air, before being refracted at the bottom at an angle of θ_a and θ_b, respectively.

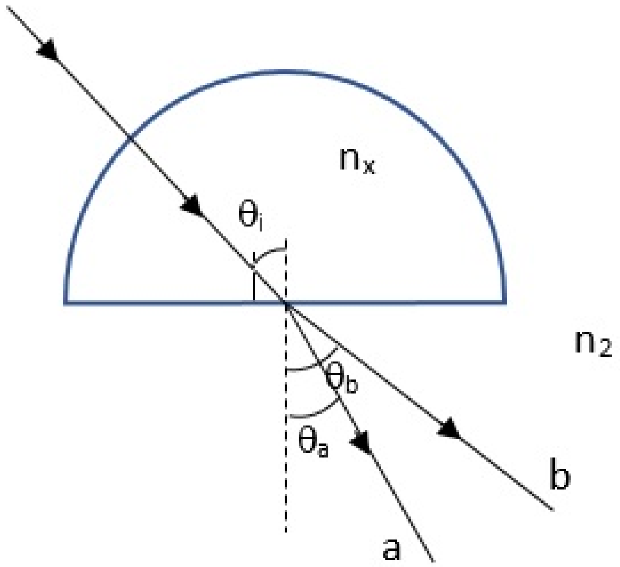

Fig. 2: Light propagation through a semi-sphere.

A.1	Which ray (a or b) propagates faster in the semi-sphere? Please provide justification.	**0.5pt**

A.2	When the incident angle, θ_i, is slowly increased to 45^o, ray b no longer exits the bottom of the semi-sphere. When θ_i is further increased to 50^o, the same happens to ray a. What is the difference between the refractive index of ray a and b in the semi-sphere?	**1.0pt**

Part B: Light Propagation Through a Cylindrical Rod (3.5 points)

A cylindrical rod has a refractive index of n_1 = 1.50. The rod is placed in air, with one end coated with a polymer with refractive index n_3 = 1.40, as shown in **Fig. 3** below. Light is incident from the polymer into the rod at an angle, θ. When θ is changed, there is an instance when light is totally reflected back to the polymer.

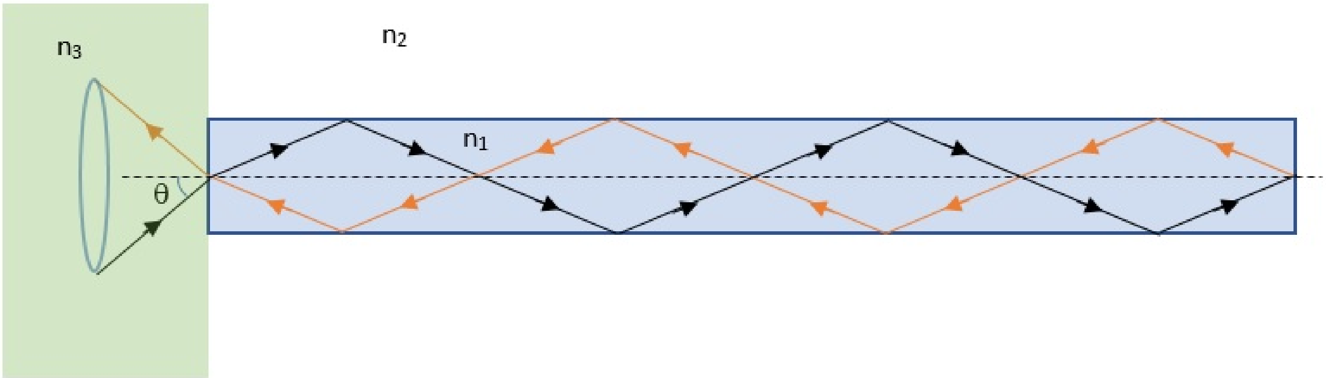

Fig. 3: A cylindrical rod with one end coated with a polymer with refractive index n_3, where $n_2 < n_3 < n_1$.

B.1	Determine the range of incident angle, θ, for this condition to happen.	**2.0pt**

B.2 How does the condition in **B.1** change,

(i) if the other, open end of the rod is now coated with a thick layer of oil with refractive index of 1.60? **0.6pt**

(ii) if the setup is placed in water with refractive index of 1.33? **0.9pt**

Part C: Light Propagation Through an Optical Fibre (5.0 points)

Optical fibre is formed by surrounding the medium with refractive index, n_1 with a lower refractive index medium, n_2, as shown in **Fig. 4** below. The medium with refractive index, n_1 is known as the fibre core, and the medium with refractive index, n_2 is known as the fibre cladding. The refractive index, n_3, is typically the refractive index of air ($n_3 = 1$).

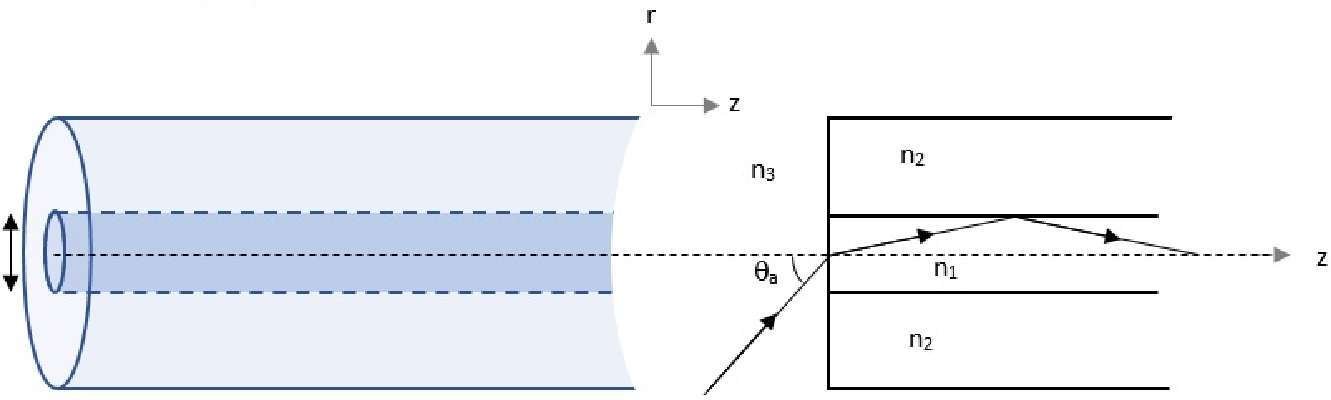

Fig. 4: A schematic of an optical fibre and its cross section.

C.1	In order for light to propagate inside the optical fiber, it first needs to be coupled into the optical fiber. The maximum angle of the light incident at the end of the fiber, θ_a so that the light can be guided inside the fiber without being refracted out is related to the refractive indices of the fiber core and cladding. Consider a ray that propagates along the meridional plane (plane that cuts across the fibre axis), formulate the relationship between the angle, θ_a and the refractive indices.	**2.0pt**

C.2	Optical fibers are not always aligned in a straight line but need to be bent to go through different spaces. Given the fiber core diameter of 50 μm, if the minimum bending radius on the optical fiber is 1.0 cm, how much will θ_a change?	**2.6pt**

C.3	Given the refractive index of the fiber core and cladding is 1.45 and 1.44, respectively, what are the maximum angle of light incident for case **C.1** and **C.2** above, when the fiber is placed in air?	**0.4pt**

The First Discovered Quasar: Unveiling the Mysteries of the Astrophysical Source 3C 273

Part A: Moon's Apparent Motion Against the Background Stars (1.8 points)

The Moon takes 27.3 days to complete 1 orbit of the Earth with respect to the stars, a period known as a sidereal month. Through a telescope, the moon's motion is readily apparent, but to the eye, it requires careful observation over several hours to notice the Moon's changing position. Remembering that 3600 arceseconds = 60 arcminutes = 1 degree then, through how many degrees, arcminutes, and arcseconds does the Moon move against the background of stars in

A.1	one hour? One minute? One second?	**1.3pt**

A.2	How long does it take the Moon to move a distance equal to its own diameter on the sky? (Note: the angular diameter of the moon as viewed from Earth is 30 arcminutes)	**0.5pt**

Part B: Using Lunar Occultations to Precisely Determine Radio Source Positions: The Case of 3C 273 (1.8 points)

In the late 1950s and early 1960s, radio position measurements with this precision were not available. While occultations had been used previously, Hazard was specifically interested in their ability to determine radio positions and structure with arcsecond precision. Such measurements were made in the optical reference frame thereby allowing reliable optical identifications. Hazard had noted that the strong Class II radio source 3C 273 would be hiddeb by the Moon (also known as occultation) several times during 1962 and 1963.

The 1962 August 5 occultation was undertaken at 410 and 136 MHz. Both disappearance and reappearance were observed and the disappearance record revealed the presence of two components, A and B, in the source. The disappearance on the right and the reappearance records on the left, at 410 MHz are plotted in **Fig. 1**.

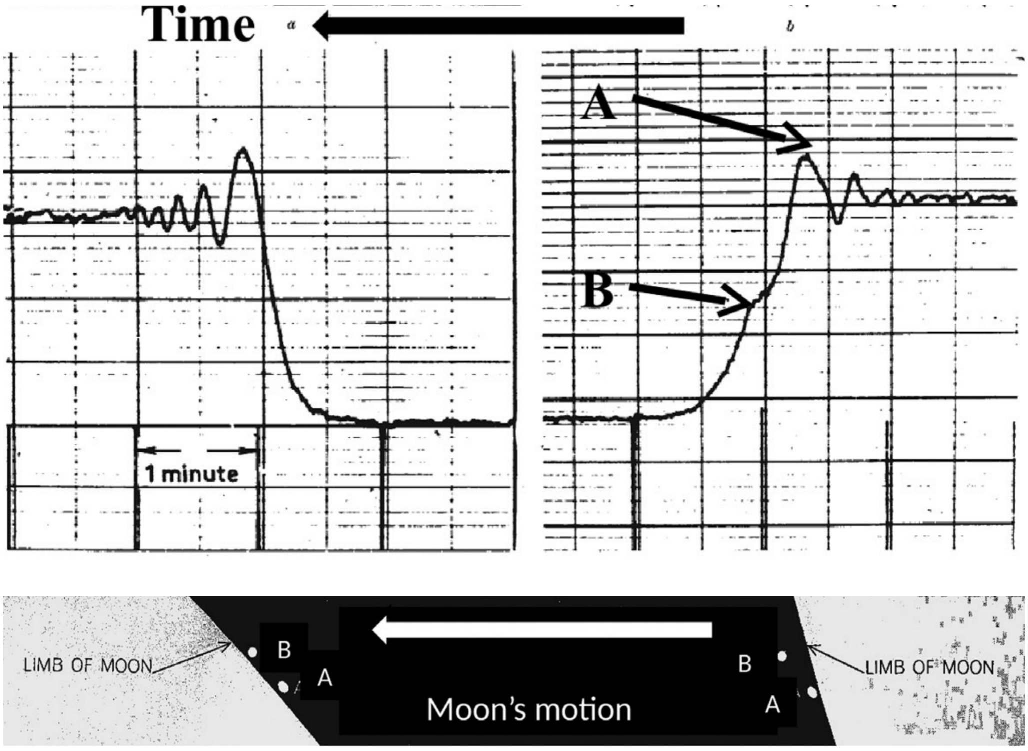

Fig. 1: The 1962 August 5 disappearance and reappearance records at 410 MHz, taken from Hazard et al. (1963). Note that time increases from right to left, and that the Moon is also moving from right to left. The bottom panel shows the positions of source components A and B relative to the limb of the Moon at disappearance and reappearance.

B.1 **0.6pt**

What is causing the oscillation in the observed intensity?

A. The diffraction pattern of the telescope

B. Instabilities in the mount of the telescope

C. The features of the surface of the Moon

D. Earth's atmospheric conditions

B.2 **0.6pt**

Why there is no extra bump for the B component in the left plot for the reappearance of the source? How does it help to understand the structure of the source?

A. The two components have been orbiting around the Moon.

B. Our line-of-sight has changed with respect to the two components.

C. Both components are aligned with the limb of the Moon when they reappear.

D. The time scales of the physics behind the luminosity of the two components were long enough to make one of them fade.

B.3 Based on your answer in **A.1** to **A.4**, and the data in **Fig. 1**, estimate the approximate apparent (projected) angular separation between the two components of 3C 273. **0.6pt**

Part C: The Breakthrough Discovery of 3C 273's True Nature (1.8 points)

In 1962, the year of these observations, Maarten Schmidt was working on the programme of optical identification and spectroscopy of the optical objects identified with radio sources. Whiteoak mentions as an afterthought that the 'current Caltech thinking' is that the potential 3C 273 identification is with a star and a strange jet. Given that no other bright star had been proposed as a radio source identification, he assumed that the bright magnitude 13 'star' (magnitude accounts as a way to measure how bright a star is; the brighter the star, the smaller the number) was merely a confusing foreground very bright star. To obtain a spectrum of the faint jet, which he saw as by far the most likely identification, it was inevitable that the bright confusing star some arcseconds away would spill over into any spectrum of the jet he would obtain. To offset this, Maarten Schmidt had decided to first obtain a spectrum of this bright star. On the night of December 29, he managed to obtain a spectrum of the bright 'star' which showed some faint emission lines (**Fig. 2**), but with no obvious explanation in terms of any expected stellar lines. Only when Schmidt decided to compare the strange spectrum with the Balmer lines of hydrogen, things became clear:

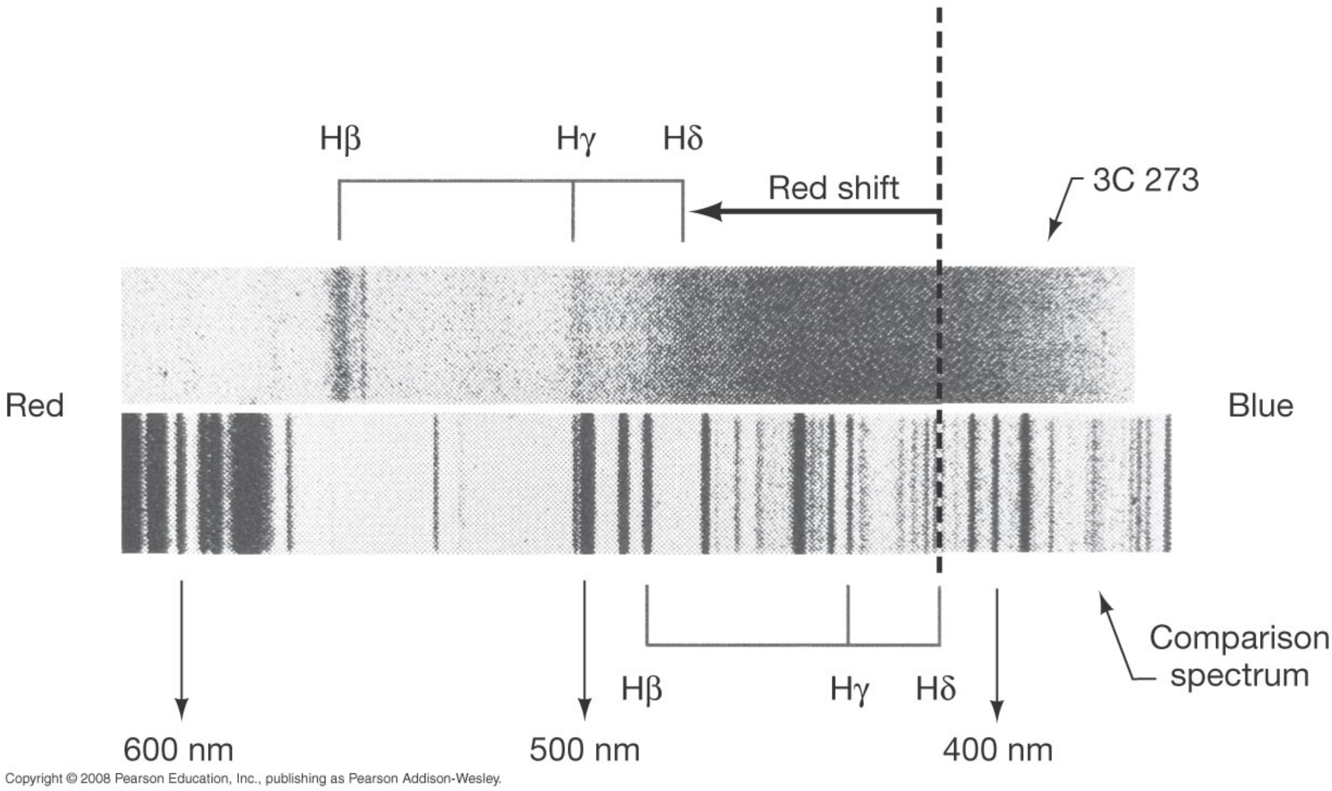

Fig. 2: Optical spectrum of 3C 273 (top) together with a comparison laboratory spectrum (bottom).

C.1	Compare the wavelength ratios of the lines shown in **Fig. 2**. Based on these ratios, what is the redshift $z = \frac{\lambda - \lambda_0}{\lambda_0}$ of this source?	**0.6pt**

C.2	Consider that this redshift is arising due to relativistic gravitational effects from the mass of the object. In this case, we can estimate the gravitational redshift as $z = \frac{1}{\sqrt{1 - \frac{2GM}{c^2 r}}} - 1$. Prove that, whether you put 3C 273 at any distance (e.g, edge of the Milky Way, r ~ 100 kpc ~ 3×10^{21} m; edge of the Solar System, r ~ 100 AU ~ 1.5×10^{13} m), the mass of 3C 273 would be so large it would disrupt the entire galaxy / solar system.	**0.6pt**

C.3	Hubble's law states that the farther the galaxies, the faster they are moving away from Earth. The Hubble's constant is the ratio between the speed and the distance of those galaxies, and equal to H=75 km/s/Mpc. With this idea in mind, we can consider instead that the calculated redshift is due to the cosmological expansion, and we can approximate that z~v/c. In that case, what is the distance of this object? Compare this with the size of the Milky Way.	**0.6pt**

Part D: The Intrinsic Luminosity of the Radio Source 3C 273 (1.8 points)

The flux of the source 3C 273 has been measure across the radio spectrum and has been found to follow the equation $F_\nu \sim 25000\nu^{(-0.3)}$ Jy, where ν is the observed frequency in Hertz and a Jansky is defined as $1\ \mathrm{Jy} = 10^{-26}\ \mathrm{W\,m^{-2}\ Hz^{-1}}$. Of course, since the source is radiating over a sphere of radius the distance from us, the intrinsic luminosity will be given by $L_\nu = 4\pi d^2 F_\nu$. With this,

D.1	What is the luminosity of the source per unit frequency? (help: $1\ \mathrm{pc} = 3.09 \times 10^{16}\ \mathrm{m}$)	**0.6pt**

D.2	What is the total luminosity in the radio band (ie., from 10^7 Hz to 10^{11} Hz)?	**0.6pt**

D.3	How does the luminosity of 3C 273 compare with the luminosity of the Sun ($L_{sun} = 3.82 \times 10^{26}$ W) and the Milky Way ($L_{MW} = 1.5 \times 10^{10} L_{sun}$) in the same frequency range?	**0.6pt**

Part E: The Power Source of 3C 273 (1.4 points)

We have seen that the brightness of 3C 273 is too large for a single star, even for an entire galaxy. Therefore, there should be another mechanism, other than stars, producing all this power.

E.1	Check that the annihilation of matter with antimatter (e.g., $e^+ + e^- \rightarrow 2\gamma$) would not explain this, since it would produce very energetic X-rays rays rather than optical and radio emission.	**0.7pt**

E.2	Something that was immediately proposed was the accretion of matter into a supermassive black hole. Let's check that potential energy can give all this energy: Consider that the black hole accretes one solar mass (2×10^{30} kg) towards its Schwarzschild radius ($R_s = \frac{2GM}{c^2}$) every year. What is the power that can be obtained for the accretion? Would this be enough to explain the observations discussed in D1 to D3?	**0.7pt**

Part F: Modern Observations and the Nature of 3C 273's Components (1.4 points)

Modern images from various telescopes (see e.g., **Fig. 3**) have found that the two components A and B that were measured with the lunar occultation actually refer to a compact core, which hosts the black hole, and a jet that extends the distance that you calculated before. This jet is thought to be produced via acceleration of the accreted particles via the strong black hole magnetic field, in a similar way particles from the Solar wind hit the magnetic field of Earth to produce the auroras near the poles, but at much larger scale.

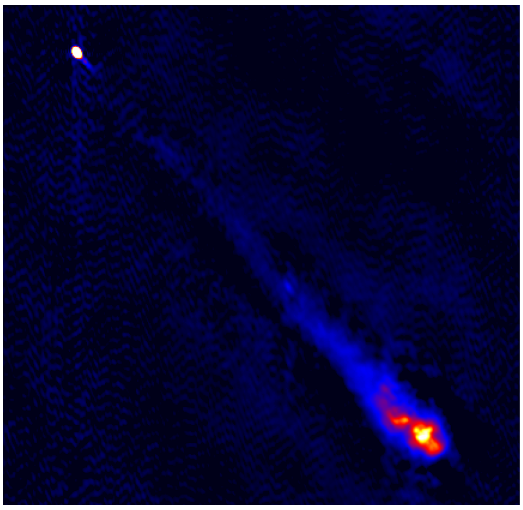

Fig. 3: Radio image of 3C 273 taken with the MERLIN telescope.

The energy density required for the production of such jet can be written in terms of the magnetic fields as the sum of the particle energy density $U_e \sim B^{(-3/2)}$ and the magnetic energy density $U_B \sim B^2$.

F.1	Show that the minimum energy density is given by $\frac{U_e}{U_B} = \frac{4}{3}$	**0.7pt**

F.2	If we assume that the total power of the 3C 273 jet is given by half of the energy released by 3C 273 in one second calculated in D.1 and D.2, and the jet has a volume of $10^{45}\ \mathrm{m}^3$, estimate the magnetic field of the jet. (hint: use that the magnetic permeability is $4\pi \times 10^{-7}\ \mathrm{T\ m\ A^{-1}}$)	**0.7pt**